PHD

Special surface classes

Pember, Mason

Award date:
2015

Awarding institution:
University of Bath

Special surface classes

submitted by

Mason James Wyndham Pember

for the degree of Doctor of Philosophy

of the

University of Bath

Department of Mathematical Sciences

February 2015

Signature of Author ....

Mason James Wyndham Pember

Summary

This thesis concerns deformations of maps into submanifolds of projective spaces and in particular the deformable surfaces of Lie sphere geometry. Using a gauge theoretic approach we study the transformations of Lie applicable surfaces and characterise certain classes of surfaces in terms of polynomial conserved quantities. In particular we unify isothermic, Guichard and L-isothermic surfaces as certain Lie applicable surfaces and show how their well known transformations arise in this setting. Another class of surfaces that is highlighted in this thesis is that of linear Weingarten surfaces in space forms and their transformations.

Acknowledgements

Firstly I would like to thank Professor Francis Burstall for his excellent supervision, his patience and the enthusiasm he showed towards my work. Our meetings have always been pleasant and insightful. I would also like express my gratitude to the geometry seminar group at the University of Bath for providing an enjoyable environment for me to pursue my research. Furthermore, I'd like to thank Dr. Daniel Clarke and Dr. Susana Santos whose well-written theses have been invaluable to me.

I am very grateful to Professor Udo Hertrich-Jeromin, who supervised me whilst he was in Bath. It was his undergraduate differential geometry course that inspired me to pursue this topic and I have learnt a great deal from our discussions. Furthermore, I am grateful for being invited to spend time in his current location, TU Wien, and the hospitality shown to me by the members of the differential geometry group there. In particular, I am grateful to Gudrun Szewieczek for our mathematical discussions and for showing me around Wien.

I'd like to say a big thank you to Professor Wayne Rossman for inviting me to Kobe University in the Summer of 2012. It was a very enjoyable, life changing experience and the knowledge I gained was invaluable. I am also very grateful to the people that I met in Japan who showed me much kindness.

I thank my examiners Professor David Calderbank and Professor Jenya Ferapontov. Thank you for your many helpful comments and for making my viva a pleasant and insightful experience.

I am thankful to my colleagues at the University of Bath. In particular, I'd like to thank the various people who have taken up residence with me in 4 West 1.15 over the past 5 years. I'd also like to say a special thank you to Ray for his efforts in organising many social events around the department. Thank you to the staff in the Department of Mathematical Sciences and thank you to EPSRC for funding this project.

I am very grateful to my family for supporting and encouraging me during my PhD. I have always been able to rely on you, especially when I felt like quitting, and thank you for driving down to Bath on numerous occasions.

I'd like to say a big thank you to my friends in Bath, especially Andra, Duncan, Jess, Jon, Jonny, Tim, Will and the Dingis family. I have been incredibly blessed to have you in my life and would not have been able to get through this without you. I'd also like to thank the leaders of Freedom Bath and Bristol church for their support and wisdom since I arrived in Bath in 2007.

Most of all I'd like to thank God - without divine intervention this thesis would never have happened! I have been very blessed over the past 5 years.

Contents

Chapter 1

Introduction

A common theme in differential geometry is the study of classes of surfaces that admit transformations that preserve their class. One of the first examples of this theme was the pseudospherical surfaces that admit a Bäcklund transformation. In fact, as we shall see, many examples of this theme admit a Bäcklund-type transformation. Bianchi discovered a famous permutability theorem for the Bäcklund transformation and it has been shown that analogous permutability theorems hold for certain Bäcklund-type transformations.

The class of isothermic surfaces is another well known example of this theme. Classically these were defined as the surfaces in Euclidean 3-space that, away from umbilic points, admit conformal curvature line coordinates. In [26], Christoffel posed the problem of characterising the pairs of surfaces in Euclidean 3-space that have parallel tangent planes and conformally equivalent induced metrics. Isothermic surfaces appear as one class of surfaces that solve this problem and this led to the discovery of the Christoffel transformation. Darboux [32] discovered a transformation for isothermic surfaces analogous to the Bäcklund transformation and Bianchi [2] showed that a permutability theorem holds for this transformation. Calapso [19] and Bianchi [1] independently developed another transformation that arises from the fact that isothermic surfaces are the deformable surfaces of conformal geometry.

Another classical example of this theme is that of Guichard surfaces [49]. These surfaces were originally characterised in Euclidean 3-space by the existence of an associate surface. This is a surface with the same spherical representation as the original surface such that a certain relation is satisfied between the principal curvature radii of the two surfaces. This yields an analogous transformation to that of the Christoffel transformation for isothermic surfaces. Calapso [20] later characterised these surfaces via the equation

$$EG(\kappa_1 - \kappa_2)^2 = E - \epsilon^2 G.$$

Eisenhart [37] developed a Bäcklund-type transformation for these surfaces and showed that a Bianchi-type permutability theorem holds for this transformation.

A lesser known example of this theme is that of L-isothermic surfaces. Blaschke [3] defined these as the surfaces in Euclidean 3-space that admit, away from umbilic points, curvature line coordinates that are conformal with respect to the third fundamental form. In recent times it has been shown that a Bäcklund-type transformation exists for these surfaces called the Bianchi-Darboux transformation [55]. Furthermore, it was shown in [57] that a transformation analogous to that discovered by Bianchi and Calapso for isothermic surfaces exists for theses surfaces. This transformation arises form the fact that L-isothermic surfaces are the deformable surfaces of Laguerre geometry [53].

Ω-surfaces are no exception to this theme. Originally discovered by Demoulin [34, 35, 36], they are characterised by the equation

$$\left(\frac{V}{U}\frac{\sqrt{E}}{\sqrt{G}}\frac{\kappa_{1,u}}{\kappa_1-\kappa_2}\right)_v+\epsilon^2\left(\frac{U}{V}\frac{\sqrt{G}}{\sqrt{E}}\frac{\kappa_{2,v}}{\kappa_1-\kappa_2}\right)_u=0 \tag{1.1}$$

given in terms of curvature line coordinates (u,v), where $\epsilon\in\{1,i\}$, U is a function of u and V is a function of v. Demoulin showed that these surfaces are the envelopes of a pair of isothermic sphere congruences and gave an alternative characterisation in terms of the existence of an associate Ω-surface, analogous to the Christoffel transformation. Eisenhart [38, 39] later developed a Bäcklund-type transformation for these surfaces. Ω_0-surfaces, the Lie geometric analogue of R_0-surfaces, are the surfaces satisfying (1.1) with $\epsilon=0$ and are envelopes of a curvature sphere congruence that is isothermic. Together, Ω- and Ω_0-surfaces constitute the applicable surfaces of Lie sphere geometry [56]. Demoulin showed that isothermic, Guichard and L-isothermic surfaces are examples of Ω-surfaces.

In [3, Section 85], Blaschke studies surfaces in Lie sphere geometry using the hexaspherical coordinate model introduced by Lie [52]. By using an adapted frame, Blaschke studies the compatibility conditions of such surfaces and in so doing finds that there are two one-forms ω_1 and ω_2 that generically determine a surface up to Lie sphere transformation. One can alternatively use the quadratic form $\omega_1\omega_2$ and the conformal class of the cubic form $\omega_1^3-\omega_2^3$. The only surfaces that are not determined by these invariants are Ω- and Ω_0-surfaces. A modern account of this is given in [42, 56].

Recent interest in integrable systems has sparked a renewed interest in Ω- and Ω_0-surfaces [42, 43, 28, 56, 14, 15, 60, 61]. In [28, Chapter 4], Clarke develops a gauge-theoretic approach for Lie applicable surfaces (and, more generally, l-applicable maps) analogous to the approach used for isothermic surfaces [18, 7, 8, 64, 11, 50], that is, they are characterised by the existence of a certain one-parameter family of flat connections. This approach lends itself well to the study of transformations of these surfaces:

- local trivialising gauge transformations of these connections give rise to a spectral deformation,
- parallel sections give rise to Bäcklund-type transformations, and
- analogues of the well known permutability theorems for transformations of isothermic surfaces [11, 50] hold for these transformations.

Furthermore, certain well known examples of Lie applicable surfaces (e.g. isothermic surfaces and Guichard surfaces) can be characterised in terms of polynomial conserved quantities of this family of flat connections.

The main focus of this thesis is the study of Ω- and Ω_0-surfaces using the approach developed in [28]. In particular, we study certain subclasses of these surfaces using polynomial conserved quantities and the resulting transformations for these subclasses. We shall see that isothermic, Guichard and L-isothermic surfaces arise from the study of linear conserved quantities, allowing us to treat these surfaces and their transformations in a unified manner. Another subclass that is given significant attention is the class of linear Weingarten surfaces, which has already received a Lie geometric treatment in [15].

In chapter 2 we review the model for Lie sphere geometry [52, 3, 25], which we will exploit throughout this thesis. In this setup one studies hypersurfaces by considering their induced Legendre maps. We shall extend the definition of Legendre map to include lines in general quadrics. We recover the two Lie geometric invariants of [3, 42]. We describe how one breaks symmetry to conformal, Laguerre and space form geometry. Furthermore, we recall the contact lift of surfaces in projective 3-space. For later use we conclude this chapter by recalling the definition of Ribaucour transformations given in [13].

In chapter 3 we study deformations of maps into projective spaces equipped with a transformation group G - a Lie group with Lie algebra $\mathfrak{g}$. We present a result that characterises G-deformable maps by the existence of a certain $\mathfrak{g}$ valued one-form. We then study the uniqueness of this one-form and characterise trivial deformations (deformations that are G-congruent to the original map). This result is then extended to G-invariant submanifolds of projective spaces. We then use these results to obtain quick proofs of known results regarding applicability and rigidity of surfaces in projective, conformal and Lie sphere geometry. In particular, we quickly recover the result of Fubini [44] that the applicability of a surface in projective 3-space is equivalent to the applicability of its contact lift. Whilst studying deformations of Legendre maps we see that there is an equivalence class of one-forms associated to each deformation. We associate a quadratic differential to this equivalence class whose vanishing determines the triviality of the deformation in question.

In chapter 4 we use the results of chapter 3 to analyse the deformable surfaces of Lie sphere geometry. The gauge theoretic approach is less straight forward than for isothermic surfaces as deforming transformations between two deformations are not unique and we have a non-trivial gauge orbit of connections to consider. This problem is solved by the introduction of the middle connection. This is a unique gauge potential in the gauge orbit that satisfies a certain property. A quadratic differential (i.e., a symmetric, traceless two-tensor with respect to a certain conformal structure) naturally arises from this setup and allows us to split the class of Lie applicable surfaces into two classes, Ω- and Ω_0-surfaces. In fact, we can give an invariant characterisation of Lie applicable surfaces in terms of this quadratic differential and the conformal class of the cubic form of [3, 42]. We show that these characterisations coincide with the classical notions of Demoulin [34, 35, 36] and recover the isothermic sphere congruences in the realm of Lie sphere geometry.

In chapter 5 we review the transformation theory of Lie applicable surfaces presented by Clarke [28] and derive these results in terms of the middle connection. This makes things clearer for subsequent chapters as we can disregard the effects of gauge transformations. In contrast to [28] we give some consideration to umbilics. For example, we see that the appearance of umbilics on Darboux transformations is attributed to the enveloping sphere congruence between the two surfaces coinciding with one of the isothermic sphere congruences.

In chapter 6 we study polynomial conserved quantities of the middle connection and in an analogous way to [18, 64] we define special Ω-surfaces of type d to be those Ω-surfaces whose middle connection admits a polynomial conserved quantity of degree d. We show that Ω-surfaces of type one project to well known surfaces in certain subgeometries of Lie sphere geometry: isothermic surfaces and Guichard surfaces in conformal geometry and L-isothermic surfaces in Laguerre geometry. The transformations of chapter 5 are shown to include the familiar transformations of these surfaces. Special Ω-surfaces of type two are then considered and are shown to project to the special Ω-surfaces of Eisenhart [39].

In chapter 7 we consider those Ω-surfaces whose middle connection admits two linear conserved quantities. These are shown to project to parallel families of non-tubular linear Weingarten surfaces in certain space forms. Furthermore tubular linear Weingarten surfaces in certain space forms are show to coincide with those Ω_0-surfaces whose middle connection admits a constant conserved quantity. The Calapso transformation of chapter 5 is shown to give a Lawson correspondence for linear Weingarten surfaces and the Darboux transformation yields a three parameter family of linear Weingarten surfaces satisfying the same linear Weingarten condition.

Chapter 8 is taken from a paper [46] written in collaboration with S. Fujimori and S. Gaber. Utilising their respective Weierstrass representations we describe a method for deforming maximal surfaces (surfaces with vanishing mean curvature) in Minkowski 3-space into their cousin constant mean curvature (CMC) 1 surfaces in de Sitter 3-space, analogous to the method of [62]. We then give new examples of genus 1 maxfaces (maximal surfaces with certain admissible singularities) in Minkowski 3-space with two or three ends to which we can apply our method to obtain their cousin genus 1 CMC 1 faces (CMC 1 surfaces with certain admissible singularities) in de Sitter 3 space.

Chapter 2

Preliminaries

In [52], Lie introduced a model for the geometry of oriented spheres in $\mathbb{R}^n \cup \{\infty\}$, i.e., Lie sphere geometry. In this setting oriented spheres are represented by points in the projective lightcone $\mathbb{P}(\mathcal{L})$ of $\mathbb{R}^{n+1,2}$ and oriented contact between two oriented spheres corresponds to the two representative points lying on a line in $\mathbb{P}(\mathcal{L})$. The transformations of this geometry are those that preserve the oriented contact of spheres and in this model the group of orthogonal transformations $\mathrm{O}(n+1,2)$ is a double cover for the group of Lie sphere transformations. Blaschke [3] developed this model further and a modern account can be found in [25].

Hypersurfaces are studied in this geometry by considering their contact lift. In the Lie sphere model this amounts to a congruence of lines in $\mathbb{P}(\mathcal{L})$. In this chapter we extend this definition to consider congruences of lines in general quadrics.

Turning to $\mathbb{R}^{4,2}$, we recall the Lie cyclide splitting used in [3] and recover the two Lie geometric invariants used in [3, 42], which we shall refer to as the Lie-invariant metric and the conformal class of the Darboux cubic form. We also show how one breaks symmetry to obtain conformal, Laguerre and space form subgeometries.

Finally, we recall the definition of Ribaucour transformations between two Legendre maps given in [13]. In Chapter 5 we shall see that the Darboux transformations of Lie applicable surfaces are of this form.

2.1 Notation

Given a vector space V and a manifold Σ, we shall denote by $\underline{V}$ the trivial bundle $\Sigma \times V$. If W is a vector subbundle of $\underline{V}$, we denote by $W^{(1)}$ the subset of $\underline{V}$ consisting of the images of sections of W and derivatives of sections of W with respect to the trivial connection on $\underline{V}$ and call $W^{(1)}$ the derived bundle of W.

Remark 2.1. *In general $W^{(1)}$ will not be a subbundle of $\underline{V}$, however, in many instances, we may assume that it is.*

Throughout this thesis we shall be considering pseudo-Euclidean spaces $\mathbb{R}^{s,t}$ where $s, t \in$

$\mathbb{N} \cup \{0\}$. These are the vector spaces $\mathbb{R}^{s+t}$ equipped with a non-degenerate symmetric bilinear form $(\,,\,)$ of signature (s,t) (see for example [58]). In the case that $s,t \geq 1$ we have that $\mathbb{R}^{s,t}$ contains lightlike vectors and we shall let $\mathcal{L}$ denote the lightcone of $\mathbb{R}^{s,t}$. The orthogonal group $\mathrm{O}(s,t)$ acts transitively on $\mathcal{L}$. We will make use of the symmetric product on $\mathbb{R}^{s,t}$: for $a,b,c \in \mathbb{R}^{s,t}$,

$$(a \odot b)c = \frac{1}{2}((a,c)b + (b,c)a).$$

Remark 2.2. *It is well known that the exterior algebra $\wedge^2\mathbb{R}^{s,t}$ is isomorphic to the Lie algebra $\mathfrak{o}(s,t)$ of $O(s,t)$, i.e., the space of skew-symmetric endomorphisms of $\mathbb{R}^{s,t}$, via the isomorphism*

$$a \wedge b \mapsto (a \wedge b)$$

where for any $c \in \mathbb{R}^{s,t}$,

$$(a \wedge b)c = (a,c)b - (b,c)a.$$

We shall make use of this identification (without warning) throughout this thesis.

Given a manifold Σ, if $\omega_1, \omega_2 \in \Omega^1(\underline{\mathbb{R}}^{s,t})$, that is ω_1 and ω_2 are one-forms on Σ with values in $\underline{\mathbb{R}}^{s,t}$, then we define $\omega_1 \curlywedge \omega_2$ to be the 2-form with values in $\wedge^2\underline{\mathbb{R}}^{s,t}$ defined by

$$\omega_1 \curlywedge \omega_2(X,Y) := \omega_1(X) \wedge \omega_2(Y) - \omega_1(Y) \wedge \omega_2(X),$$

for $X, Y \in \Gamma T\Sigma$. Notice that $\omega_1 \curlywedge \omega_2 = \omega_2 \curlywedge \omega_1$.

2.2 Legendre maps

Suppose that $s,t \in \mathbb{N}$ such that $s,t \geq 2$. Then $\mathbb{R}^{s,t}$ contains isotropic two dimensional subspaces and we shall let $\mathcal{Z}$ denote the Grassmannian of isotropic two dimensional subspaces of $\mathbb{R}^{s,t}$.

Remark 2.3. *Throughout this thesis we will make use of the identification of $\mathcal{Z}$ with the space of lines in $\mathbb{P}(\mathcal{L})$.*

Let $n := s + t - 4$ and suppose that Σ is an n-dimensional manifold. Let $f : \Sigma \to \mathcal{Z}$ be a smooth map. Then we may define a tensor, analogous to the solder form defined in [7, 10, 17],

$$\beta : T\Sigma \to Hom(f, f^{(1)}/f), \quad X \mapsto (\sigma \mapsto d_X\sigma \, mod\, f),$$

where here we are viewing f as a rank two subbundle of the trivial bundle over Σ,

$$\underline{\mathbb{R}}^{s,t} := \Sigma \times \mathbb{R}^{s,t}.$$

The following definition generalises Pinkall's notion of Lie geometric hypersurfaces given in [59], which Cecil [25] referred to as Legendre submanifolds, where it was assumed that $t = 2$:

Definition 2.4. *We say that f is a Legendre map if the contact condition $f^{(1)} = f^{\perp}$ is satisfied and the immersion condition $\ker\beta = \{0\}$ holds.*

Remark 2.5. *$f^{\perp}/f$ is a rank n subbundle of $\underline{\mathbb{R}}^{s,t}/f$ whose induced metric is non-degenerate with signature $(s-2, t-2)$.*

2.2.1 Curvature spheres

Suppose that $f : \Sigma \to \mathcal{Z}$ is a Legendre map and let $f^{\mathbb{C}} := f \otimes \mathbb{C}$.

Definition 2.6. *Let $p \in \Sigma$. Then a one dimensional subspace $s(p) \leq f^{\mathbb{C}}(p)$ is a curvature sphere of f at p if there exists a non-zero subspace $T^{s(p)} \leq T_p\Sigma \otimes \mathbb{C}$ such that $\beta(T^{s(p)})s(p) = 0$. We call the maximal such $T^{s(p)}$ the curvature space of $s(p)$.*

A rank one $s \leq f^{\mathbb{C}}$ is called a curvature sphere congruence of f if at each point $p \in \Sigma$, $s(p)$ is a curvature sphere of f at p. We denote by T^s the subset of $T\Sigma \otimes \mathbb{C}$ formed by the congruence over Σ of curvature spaces $T^{s(p)}$.

Remark 2.7. *In general T^s will not be a subbundle of $T\Sigma \otimes \mathbb{C}$ as the dimension of $T^{s(p)}$ may not be constant over Σ.*

Remark 2.8. *Definition 2.6 coincides with the definition given in [25, Section 4.4] since $\beta(T^{s(p)})s(p) = 0$ is equivalent to*

$$d_X\sigma \in f^{\mathbb{C}}(p),$$

for all $\sigma \in \Gamma s$ and $X \in T^{s(p)}$.

Let $s \leq f$ be a rank one subbundle of f such that $\ker \beta_p s = \{0\}$ for all $p \in \Sigma$. Let $\sigma \in \Gamma s$ be a lift of s and let $\tilde{\sigma} \in \Gamma f$ be linearly independent everywhere to σ. Then $\beta\sigma$ is an isomorphism from $T\Sigma$ to $f^{\perp}/f$ and there exists $A \in \Gamma End(T\Sigma)$ such that

$$\beta\tilde{\sigma} = (\beta \circ A)\sigma.$$

Therefore,

$$d\tilde{\sigma} = d\sigma \circ A \, mod \, f$$

and the closure of $d\tilde{\sigma}$ implies that A is symmetric with respect to the non-degenerate metric $(d\sigma, d\sigma)$. Let $\lambda_{1,p}, ..., \lambda_{r,p} \in \mathbb{C}$ be the distinct eigenvalues of A_p and let $T_{1,p}, ..., T_{r,p} \leq T_p\Sigma \otimes \mathbb{C}$ be the corresponding eigenspaces of $T_p\Sigma \otimes \mathbb{C}$. Then

$$\beta(T_{i,p})(\tilde{\sigma} - \lambda_{i,p}\sigma) = 0.$$

Hence, $s_i(p) := \langle \tilde{\sigma}(p) - \lambda_{i,p}\sigma(p) \rangle \leq f^{\mathbb{C}}(p)$ is a curvature sphere of f at p with curvature space $T_{i,p}$.

Proposition 2.9. *Suppose that $t = 2$. Then the curvature spheres $s_1(p), ..., s_r(p)$ of f are real and*

$$T_p\Sigma = T_{1,p} \oplus ... \oplus T_{r,p}.$$

Proof. By Remark 2.5, the induced metric on $f^{\perp}/f$ is positive definite. Hence, $(d\sigma, d\sigma)$ is a positive definite metric. Since A is symmetric with respect to $(d\sigma, d\sigma)$ we have that A_p

is diagonalisable over $\mathbb{R}$. Therefore all the eigenvalues $\lambda_{1,p}, ..., \lambda_{r,p}$ of A_p are real and thus $s_{1,p}, ..., s_{r,p}$ are real and the curvature spaces span $T_p\Sigma$. $\square$

Remark 2.10. *From now on we shall assume that, for all $p \in \Sigma$, A_p is diagonalisable over $\mathbb{C}$ and thus*

$$T_p\Sigma \otimes \mathbb{C} = (T_{1,p} \oplus ... \oplus T_{r,p}) \otimes \mathbb{C}.$$

Definition 2.11. *We say that f is umbilic at a point $p \in \Sigma$ if f has exactly one curvature sphere at p. Moreover, if f has only one curvature sphere congruence then we say that f is totally umbilic.*

Remark 2.12. *If f is umbilic at a point p, then the corresponding curvature sphere $s(p)$ has to be real.*

Lemma 2.13. *Suppose that Σ is simply connected. Then f is totally umbilic if and only if there exists a non-zero $\mathfrak{q} \in \mathbb{R}^{s,t}$ such that $\mathfrak{q} \in \Gamma f$.*

Proof. If $\mathfrak{q} \in \mathbb{R}^{s,t}$ then $d\mathfrak{q} = 0$ and therefore $\mathfrak{q} \in \Gamma f$ implies that $s := \langle \mathfrak{q} \rangle$ is a curvature sphere congruence of f with curvature subbundle $T^s = T\Sigma$. By the immersion condition of f this is the only curvature sphere congruence of f and thus f is totally umbilic.

Conversely, suppose that s is the only curvature sphere congruence of f. Then for any lift $\sigma \in \Gamma s$ and linearly independent $\tilde{\sigma} \in \Gamma f$ we have that

$$d\sigma = \omega\,\sigma + \tilde{\omega}\,\tilde{\sigma}.$$

The closure of $d\sigma$ implies that

$$0 = d\omega\,\sigma - \omega \wedge \tilde{\omega}\,\tilde{\sigma} + d\tilde{\omega}\,\tilde{\sigma} - \tilde{\omega} \wedge d\tilde{\sigma}.$$

Now since s is the only curvature sphere congruence of f, we must have that $d\tilde{\sigma}$ never belongs to $\Omega^1(f)$. Hence, $\tilde{\omega} = 0$ and $d\omega = 0$. Since Σ is simply connected we may choose a non-zero function μ such that $d\mu = -\mu\,\omega$ and then

$$d(\mu\sigma) = d\mu\,\sigma + \mu\,\omega\,\sigma = 0.$$

Hence, $\mu\sigma \in \Gamma f$ is constant. $\square$

Suppose that f has $r \leq n$ distinct curvature sphere congruences $s_1, ..., s_r \leq f^{\mathbb{C}}$ and corresponding curvature subbundles[1] $T_1, ..., T_r \leq T\Sigma \otimes \mathbb{C}$. Then for each $i \in \{1, ..., r\}$ let f_i be the subbundle of $(f^{\mathbb{C}})^{\perp}$ of sections of $f^{\mathbb{C}}$ and derivatives of sections of $f^{\mathbb{C}}$ along T_i. We shall call f_i the derived bundle of f along T_i.

[1] Notice that here we are assuming that the dimension of the curvature space of each curvature sphere is constant over Σ.

Lemma 2.14. *For any $\sigma \in \Gamma f^{\mathbb{C}}$ such that σ never belongs to s_i, we have that*

$$f_i = f^{\mathbb{C}} \oplus d\sigma(T_i).$$

Furthermore, $f_i \perp f_j$ for $i \neq j$ and

$$(f^{\mathbb{C}})^{\perp}/f^{\mathbb{C}} = f_1/f^{\mathbb{C}} \oplus ... \oplus f_r/f^{\mathbb{C}}$$

with the induced metric on each $f_i/f^{\mathbb{C}}$ being non-degenerate.

Proof. Firstly, if $\sigma \in \Gamma f^{\mathbb{C}}$ never belongs to s_i, then for all $X \in \Gamma T_i$, $d_X\sigma$ never belongs to $f^{\mathbb{C}}$ by the immersion condition of f. Therefore, it makes sense to write

$$f^{\mathbb{C}} \oplus d\sigma(T_i).$$

For any other section $\nu \in \Gamma f^{\mathbb{C}}$, we may write

$$\nu = \lambda\sigma + \sigma_i,$$

for some smooth function λ and $\sigma_i \in \Gamma s_i$. Therefore,

$$d_X\nu = \lambda d_X\sigma \, mod \, f^{\mathbb{C}},$$

for any $X \in \Gamma T_i$. Hence,

$$f_i = f^{\mathbb{C}} \oplus d\sigma(T_i).$$

Now suppose that $i, j \in \{1, ..., r\}$ such that $i \neq j$ and let $v \in \Gamma f_i$ and $w \in \Gamma f_j$. Then there exist lifts $\sigma_i \in \Gamma s_i$ and $\sigma_j \in \Gamma s_j$ such that

$$v = d_X\sigma_j + \xi_1 \quad \text{and} \quad w = d_Y\sigma_i + \xi_2,$$

for some $X \in \Gamma T_i$, $Y \in \Gamma T_j$ and $\xi_1, \xi_2 \in \Gamma f^{\mathbb{C}}$. Thus,

$$(v, w) = (d_X\sigma_j, d_Y\sigma_i) = d_X(\sigma_j, d_Y\sigma_i) - (\sigma_j, d_X d_Y\sigma_i) = -(\sigma_j, d_X d_Y\sigma_i),$$

since $f \leq f^{\perp}$. Now, using that d is flat we have that

$$(v, w) = -(\sigma_j, d_Y d_X\sigma_i) - (\sigma_j, d_{[X,Y]}\sigma_i).$$

It then follows from the fact that $d_X\sigma_i \in \Gamma f^{\mathbb{C}}$ that $d_Y d_X\sigma_i \in \Gamma(f^{(1)} \otimes \mathbb{C})$. By the contact condition $f^{(1)} = f^{\perp}$, we then have that $(v, w) = 0$. Since $v \in \Gamma f_i$ and $w \in \Gamma f_j$ were arbitrary, we have that $f_i \perp f_j$.

Using the contact condition $f^{(1)} = f^{\perp}$ we have that $f^{\perp}/f = f^{(1)}/f$ and thus

$$(f^{\mathbb{C}})^{\perp}/f^{\mathbb{C}} = f_1/f^{\mathbb{C}} + ... + f_r/f^{\mathbb{C}}.$$

Now if for some $i \in \{1, ..., r\}$ the induced metric on $f_i/f^{\mathbb{C}}$ were degenerate then, since $f_i \perp f_j$ for all $j \neq i$, so would the induced metric on $(f^{\mathbb{C}})^{\perp}/f^{\mathbb{C}}$. This would then contradict Remark 2.5. Hence, the induced metric on f_i/f is non-degenerate and

$$(f^{\mathbb{C}})^{\perp}/f^{\mathbb{C}} = f_1/f^{\mathbb{C}} \oplus ... \oplus f_r/f^{\mathbb{C}}.$$

□

Remark 2.15. *In the case that all the curvature sphere congruences are real, we may (and will) drop the "$\mathbb{C}$" from Lemma 2.14.*

The following lemma is a generalisation of a result of U. Pinkall [59, Proposition 2] (see also [25, 63]):

Lemma 2.16. *Each T_i is integrable. Furthermore, if for some $i \in \{1, ..., r\}$, $rank T_i > 1$, then s_i is constant along the leaves of T_i.*

Proof. Let $X, Y \in \Gamma T_i$ and $\sigma_i \in \Gamma s_i$. Then, since d is flat, we have that

$$d_{[X,Y]}\sigma_i = d_X d_Y \sigma_i - d_Y d_X \sigma_i. \tag{2.1}$$

Now, since $d_X\sigma_i, d_Y\sigma_i \in \Gamma f^{\mathbb{C}}$, we have that the right hand side of (2.1) takes values in f_i. Therefore, $d_{[X,Y]}\sigma_i \in \Gamma f_i$ and by Lemma 2.14, $[X, Y] \in \Gamma T_i$. Hence, T_i is integrable.

Suppose that $\operatorname{rank} T_i > 1$ and let $X, Y \in \Gamma T_i$ be linearly independent. Let σ_i be a lift of s_i and let $\tilde{\sigma} \in \Gamma f^{\mathbb{C}}$ such that $\tilde{\sigma}$ never belongs to s_i. Then

$$d_X\sigma_i = \alpha\sigma_i + \beta\tilde{\sigma} \quad \text{and} \quad d_Y\sigma_i = \gamma\sigma_i + \delta\tilde{\sigma},$$

for some smooth functions α, β, γ and δ. Since d is flat and T_i is integrable, we have that

$$0 \quad = \quad (d_X d_Y \sigma_i - d_Y d_X \sigma_i - d_{[X,Y]}\sigma_i) \, mod \, f = d_{\delta X - \beta Y} \tilde{\sigma} \, mod \, f.$$

Now, since $\tilde{\sigma}$ never belongs to s_i, the immersion condition of f implies that $\delta X - \beta Y = 0$. The linear independence of X and Y then implies that β and δ vanish. Thus,

$$d|_{T_i}\sigma_i \in \Gamma(T_i^* \otimes s_i).$$

Therefore, s_i is constant along the leaves of T_i. □

2.3 Dupin cyclides

Let $f : \Sigma \to Z$ be a Legendre map with curvature spheres $s_1, ..., s_r \leq f^{\mathbb{C}}$, with corresponding curvature subbundles $T_1, ..., T_r \leq T\Sigma \otimes \mathbb{C}$. For $i \in \{1, ..., r\}$ we define maps

$$\beta_i : T_i \to Hom(s_i, f^{\mathbb{C}}/s_i), \quad X \mapsto (\sigma_i \mapsto d_X\sigma_i \, mod \, s_i).$$

Analogously to [59], we make the following definition:

Definition 2.17. *f is a Dupin cyclide if f has exactly two curvature spheres s_1, s_2 everywhere and s_i is constant along the leaves of T_i for each $i \in \{1, 2\}$, i.e.,*

$$\beta_1 = \beta_2 = 0.$$

Remark 2.18. *By Lemma 2.16, if f has exactly two curvature spheres everywhere and their curvature subbundles both have rank greater than one, then f is a Dupin cyclide.*

2.3.1 Lie cyclides

Now suppose that $(s, t) = (4, 2)$ and suppose that $f : \Sigma \to \mathcal{Z}$ is an umbilic-free Legendre map. In [3], Blaschke defined the Lie cyclides of a Legendre map. These are the congruence of Dupin cyclides that make most contact with the Legendre map at each point. This notion yields a useful splitting of the trivial bundle $\underline{\mathbb{R}}^{4,2}$, which we shall use in Chapter 4.

Let $\sigma_1 \in \Gamma s_1$ and $\sigma_2 \in \Gamma s_2$ be lifts of the curvature sphere congruences and let X be a lift of T_1 and Y be a lift of T_2. Then

$$d_X\sigma_1, d_Y\sigma_2 \in \Gamma f.$$

Let

$$S_1 := \langle \sigma_1, d_Y\sigma_1, d_Y d_Y\sigma_1 \rangle \quad \text{and} \quad S_2 := \langle \sigma_2, d_X\sigma_2, d_X d_X\sigma_2 \rangle .$$

Proposition 2.19 ([3])**.** *S_1 and S_2 are orthogonal rank 3 subbundles of $\underline{\mathbb{R}}^{4,2}$ and the induced metric on each S_i has signature $(2, 1)$. We then have the orthogonal splitting*

$$\underline{\mathbb{R}}^{4,2} = S_1 \oplus_\perp S_2.$$

Furthermore, S_1 and S_2 do not depend on choices.

This splitting now yields a splitting of the trivial connection d on $\underline{\mathbb{R}}^{4,2}$:

$$d = \mathcal{D} + \mathcal{N},$$

where $\mathcal{D}$ is the direct sum of the induced connections on S_1 and S_2 and

$$\mathcal{N} = d - \mathcal{D} \in \Omega^1((Hom(S_1, S_2) \oplus Hom(S_2, S_1)) \cap \mathfrak{o}(4, 2)).$$

Since S_1 and S_2 are orthogonal, we have that $\mathcal{D}$ is a metric connection on $\underline{\mathbb{R}}^{4,2}$ and $\mathcal{N}$ is a skew-symmetric endomorphism. Hence, $\mathcal{N} \in \Omega^1(S_1 \wedge S_2)$.

Lemma 2.20. *$\mathcal{N} f \leq \Omega^1(f)$ and*

$$\mathcal{N}(T_2)s_1 = 0 = \mathcal{N}(T_1)s_2.$$

Furthermore, $\beta_i = \mathcal{N}|_{s_i} \, mod \, s_i$.

Proof. Suppose that $\sigma_1 \in \Gamma s_1$. Then for any $Y \in \Gamma T_2$, $d_Y\sigma_1 \in \Gamma S_1$ and thus $\mathcal{N}_Y\sigma_1 = 0$. Furthermore, since s_1 is a curvature sphere, $d_X\sigma_1 \in \Gamma f$. Hence, $\mathcal{N}s_1 \leq \Omega(f)$. Moreover,

$$\mathcal{N}_X\sigma_1 \, mod \, s_1 = (d_X\sigma_1 - \mathcal{D}_X\sigma_1) \, mod \, s_1 = d_X\sigma_1 \, mod \, s_1 = \beta_1(X)\sigma_1.$$

Hence, $\beta_1 = \mathcal{N}|_{s_1} \, mod \, s_1$. An analogous argument can be used for $i = 2$. $\square$

We immediately obtain the following corollaries:

Corollary 2.21. *s_i is constant along the leaves of T_i for some $i \in \{1, 2\}$ if and only if $\mathcal{N}|_{s_i} = 0$.*

Corollary 2.22. *f is a Dupin cyclide if and only if $\mathcal{N}|_f \equiv 0$.*

2.4 Symmetry breaking

Assume that $(s, t) = (4, 2)$. In [25] a modern account is given of how one breaks symmetry from Lie geometry to space-form geometry and how $\mathrm{O}(4, 2)$ is a double cover for the group of Lie sphere transformations. These are the transformations that map oriented spheres to oriented spheres and preserve the oriented contact of spheres. In this section we shall recall the process of symmetry breaking.

Lemma 2.23. *Suppose that $f : \Sigma \to \mathcal{Z}$ is a Legendre map and $\mathfrak{q} \in \mathbb{R}^{4,2}\backslash\{0\}$. Then*

1. *if $\mathfrak{q}$ is timelike then f never belongs to $\langle \mathfrak{q} \rangle^\perp$,*
2. *if $\mathfrak{q}$ is spacelike then the set of points $p \in \Sigma$ where $f(p) \leq \langle \mathfrak{q} \rangle^\perp$ is a closed set with empty interior,*
3. *$f \leq \langle \mathfrak{q} \rangle^\perp$ if and only if $\mathfrak{q} \in \Gamma f$, in which case f is totally umbilic.*

Proof. If $\mathfrak{q}$ is timelike then $\langle \mathfrak{q} \rangle^\perp$ has signature $(4, 1)$ and the maximal lightlike subspaces of $\langle \mathfrak{q} \rangle^\perp$ are one-dimensional. Therefore since $f(p)$ is a two dimensional lightlike subspace for each $p \in \Sigma$, $f(p) \nleq \langle \mathfrak{q} \rangle^\perp$.

Suppose that $\mathfrak{q}$ is spacelike and that on some open subset $U \subset \Sigma$, $f \leq \langle \mathfrak{q} \rangle^\perp$. Without loss of generality, assume that $U = \Sigma$. Then this implies that $f^{(1)} \leq \langle \mathfrak{q} \rangle^\perp$ and $\mathfrak{q} \in \Gamma f^\perp$. Hence, $f^{(1)} \neq f^\perp$, contradicting the contact condition of f.

Therefore, if $f \leq \langle \mathfrak{q} \rangle^\perp$ then the only possibility left to consider is that $\mathfrak{q}$ is lightlike. Then since the maximal lightlike subspaces of $\mathbb{R}^{4,2}$ are two dimensional, $\mathfrak{q} \in \Gamma f^\perp$ if and only if $\mathfrak{q} \in \Gamma f$. By Lemma 2.13 this is the case only if f is totally umbilic. $\square$

We shall often refer to a non-zero vector $\mathfrak{q} \in \mathbb{R}^{4,2}$ as a sphere complex. As Lemma 2.23 shows, for a Legendre map $f : \Sigma \to \mathcal{Z}$, generically $f \cap \langle \mathfrak{q} \rangle^\perp$ defines a rank one subbundle of f.

2.4.1 Conformal geometry

Let $\mathfrak{p} \in \mathbb{R}^{4,2}$ such that $\mathfrak{p}$ is not lightlike. If $\mathfrak{p}$ is timelike then $\langle \mathfrak{p} \rangle^\perp \cong \mathbb{R}^{4,1}$ and defines a Riemannian conformal geometry. If $\mathfrak{p}$ is spacelike then $\langle \mathfrak{p} \rangle^\perp \cong \mathbb{R}^{3,2}$ and defines a Lorentzian conformal geometry. We consider elements of

$$\mathbb{P}(\mathcal{L} \cap \langle \mathfrak{p} \rangle^\perp)$$

to be points and refer to $\mathfrak{p}$ as a point sphere complex.

Remark 2.24. *In the case that $\mathfrak{p}$ is timelike, we have that $\mathbb{P}(\mathcal{L} \cap \langle \mathfrak{p} \rangle^\perp)$ is the conformal 3-sphere (see [50]).*

The elements of $\mathbb{P}(\mathcal{L} \backslash \langle \mathfrak{p} \rangle^\perp)$ give rise to spheres in the following way: suppose that $s \in \mathbb{P}(\mathcal{L} \backslash \langle \mathfrak{p} \rangle^\perp)$. Now $s \oplus \langle \mathfrak{p} \rangle$ is a $(1,1)$-plane and thus

$$V := (s \oplus \langle \mathfrak{p} \rangle)^\perp$$

is a $(3,1)$-plane. The projective lightcone of V is then diffeomorphic to $\mathbb{S}^2$ and we thus identify V with a sphere in $\mathbb{P}(\mathcal{L} \cap \langle \mathfrak{p} \rangle^\perp)$.

Conversely, suppose that $V \leq \langle \mathfrak{p} \rangle^\perp$ is a $(3,1)$-plane. Then $V^\perp$ is a $(1,1)$-plane in $\mathbb{R}^{4,2}$ containing $\mathfrak{p}$ and we identify the two null lines of $V^\perp$ with the sphere defined by V with opposite orientations.

Remark 2.25. *Those Lie sphere transformations that fix the point sphere complex are the conformal transformations of $\langle \mathfrak{p} \rangle^\perp$.*

Space form geometry

As is standard in conformal geometry (see, for example, [50]), we may break symmetry further by choosing a vector $\mathfrak{q} \in \langle \mathfrak{p} \rangle^\perp$. Then

$$\mathfrak{Q}^3 := \{y \in \mathcal{L} : \, (y, \mathfrak{q}) = -1, \, (y, \mathfrak{p}) = 0\}$$

is isometric to a space form with sectional curvature $\kappa = -|\mathfrak{q}|^2$. If we assume that $|\mathfrak{p}|^2 = \pm 1$, then

$$\mathfrak{P}^3 := \{y \in \mathcal{L} : \, (y, \mathfrak{q}) = 0, \, (y, \mathfrak{p}) = -1\}$$

can be identified (see [50]) with the space of hyperplanes (complete, totally geodesic hypersurfaces) in this space form.

Suppose that $f : \Sigma \to \mathcal{Z}$ is a Legendre map. Then, by Lemma 2.23, on a dense open subset of Σ, $\Lambda := f \cap \langle \mathfrak{p} \rangle^\perp$ is rank one subbundle of f. Using the identification of $\wedge^2 \mathbb{R}^{4,2}$ with the skew-symmetric endomorphisms on $\mathbb{R}^{4,2}$, we have for any $\tau \in \Gamma \wedge^2 f$ that $\tau \mathfrak{p} \in \Gamma f$ and, since τ is skew-symmetric, $\tau \mathfrak{p} \perp \mathfrak{p}$. Hence,

$$\Lambda = (\wedge^2 f)\mathfrak{p}.$$

Away from points where $\Lambda \perp \mathfrak{q}$, we have that for any $\tau \in \Gamma(\wedge^2 f)$

$$\mathfrak{f} := -\frac{\tau\mathfrak{p}}{(\tau\mathfrak{p},\mathfrak{q})} \quad \text{and} \quad \mathfrak{t} := -\frac{\tau\mathfrak{q}}{(\tau\mathfrak{q},\mathfrak{p})}$$

are the projections of f into $\mathfrak{Q}^3$ and $\mathfrak{P}^3$, respectively. We can then write $f = \langle \mathfrak{f}, \mathfrak{t}\rangle$.

Definition 2.26. *We call* $\mathfrak{f}$ *the space form projection of* f *and* $\mathfrak{t}$ *the tangent plane*[2] *congruence of* f.

One can easily see that:

Lemma 2.27. *The space form projection of* f *into* $\mathfrak{Q}^3$ *exists at* $p \in \Sigma$ *if and only if the kernel of the linear map*

$$\wedge^2 f \to \mathbb{R}, \quad \tau \mapsto (\tau\mathfrak{p},\mathfrak{q})$$

is trivial at p.

Away from umbilic points, suppose that (u, v) are curvature line coordinates for $\mathfrak{f}$. Then by Rodrigues' equations we have that

$$\mathfrak{t}_u + \kappa_1 \mathfrak{f}_u = 0 = \mathfrak{t}_v + \kappa_1 \mathfrak{f}_v,$$

where κ_1 and κ_2 are the principal curvatures of $\mathfrak{f}$. Therefore,

$$s_1 := \langle \mathfrak{t} + \kappa_1 \mathfrak{f}\rangle \quad \text{and} \quad s_2 := \langle \mathfrak{t} + \kappa_2 \mathfrak{f}\rangle$$

are curvature spheres of f with respective curvature subbundles $T_1 := \langle \frac{\partial}{\partial u}\rangle$ and $T_2 := \langle \frac{\partial}{\partial v}\rangle$.

2.4.2 Laguerre geometry

In this subsection we shall recall the correspondence given in [25] between Lie sphere geometry and Laguerre geometry. Let $\mathfrak{q}_\infty \in \mathcal{L}$ and define $U := \mathbb{P}(\mathcal{L})\backslash\langle \mathfrak{q}_\infty\rangle^\perp$. Then (E, ψ) with

$$E := \{y \in \mathcal{L} : (y, \mathfrak{q}_\infty) = -1\} \quad \text{and} \quad \psi : E \to U, \quad y \mapsto [y]$$

defines an affine chart for U. Choosing $\mathfrak{q}_0 \in \mathcal{L}$ such that $(\mathfrak{q}_0, \mathfrak{q}_\infty) = -1$, we have that $\langle \mathfrak{q}_0, \mathfrak{q}_\infty\rangle^\perp \cong \mathbb{R}^{3,1}$. We may then define the orthogonal projection

$$\pi : \mathbb{R}^{4,2} \to \langle \mathfrak{q}_0, \mathfrak{q}_\infty\rangle^\perp, \quad y \mapsto y + (y, \mathfrak{q}_\infty)\mathfrak{q}_0 + (y, \mathfrak{q}_0)\mathfrak{q}_\infty.$$

Then $\pi \circ \psi^{-1}$ defines an isomorphism between U and $\langle \mathfrak{q}_0, \mathfrak{q}_\infty\rangle^\perp$. We thus identify points in U as points in $\mathbb{R}^{3,1}$. Now let $W := \mathbb{P}(\mathcal{L} \cap \langle \mathfrak{q}_\infty\rangle^\perp)\backslash\langle \mathfrak{q}_\infty\rangle$. Then π identifies W with the projective lightcone of $\langle \mathfrak{q}_0, \mathfrak{q}_\infty\rangle^\perp$ and thus $\mathbb{P}(\mathcal{L}^3)$, where $\mathcal{L}^3$ is the lightcone of $\mathbb{R}^{3,1}$. Therefore, we identify W with null directions in $\mathbb{R}^{3,1}$. We define $\langle \mathfrak{q}_\infty\rangle$ to be the improper point of Laguerre geometry.

[2] Note that "plane" here means totally geodesic hypersurface in the space form $\mathfrak{Q}^3$.

Under this correspondence, contact elements in $\mathbb{R}^{4,2}$ are then identified with affine null lines in $\mathbb{R}^{3,1}$, i.e., for $z \in \mathbb{R}^{3,1}$ and $l \in \mathbb{P}(\mathcal{L}^3)$

$$L = [z, l] := \{z + v : v \in l\}.$$

Euclidean projection

By choosing a point sphere complex $\mathfrak{p} \in \langle \mathfrak{q}_0, \mathfrak{q}_\infty \rangle^\perp$ with $|\mathfrak{p}|^2 = -1$, we have that

$$\langle \mathfrak{q}_0, \mathfrak{q}_\infty, \mathfrak{p} \rangle^\perp \cong \mathbb{R}^3.$$

Using isotropy projection [3, 25], one identifies points in $\mathbb{R}^{3,1}$ with oriented spheres (including point spheres, but not oriented planes) in $\mathbb{R}^3$: a sphere centred at $c \in \mathbb{R}^3$ with signed radius $r \in \mathbb{R}$ is identified with the point

$$c + r\mathfrak{p} \in \mathbb{R}^{3,1}.$$

We then have that null lines in $\mathbb{R}^{3,1}$ correspond to pencils of spheres in $\mathbb{R}^3$ in oriented contact with each other and isotropic planes in $\mathbb{R}^{3,1}$ are identified with oriented planes in $\mathbb{R}^3$.

It was shown in [25] that the Lie sphere transformations $A \in \mathrm{O}(4,2)$ that preserve the improper point $\langle \mathfrak{q}_\infty \rangle$ are identified under this correspondence with the affine Laguerre transformations of $\mathbb{R}^{3,1}$, that is, the identity component of the group $\mathbb{R}^4 \rtimes \mathrm{O}(3,1)$. In terms of transformations of $\mathbb{R}^3$, this group consists of the Lie sphere transformations that map oriented planes to oriented planes.

Defining

$$\mathfrak{Q}^3 := \{y \in \mathcal{L} : (y, \mathfrak{q}_\infty) = -1, (y, \mathfrak{p}) = 0\},$$

we have that $\pi|_{\mathfrak{Q}^3}$ is an isometry between $\mathfrak{Q}^3$ and $\langle \mathfrak{q}_0, \mathfrak{q}_\infty, \mathfrak{p} \rangle^\perp$ and this restricts to the usual Euclidean projection in the conformal geometry defined by $\langle \mathfrak{p} \rangle^\perp$, see [18, 64, 50].

2.5 Invariants of Lie sphere geometry

Suppose that $(s,t) = (4,2)$. Let $f : \Sigma \to \mathcal{Z}$ be a Legendre map. We will now recover the Lie-invariant metric and conformal class of the cubic form used in [3, 42]. These invariants generically[3] determine a surface up to Lie sphere transformation. Let $f : \Sigma \to \mathcal{Z}$ be a Legendre map.

2.5.1 Conformal structure

Define a tensor $c \in \Gamma(S^2 T^*\Sigma \otimes (\wedge^2 f)^* \otimes \wedge^2 (f^\perp / f))$ by

$$c(X,Y)\xi_1 \wedge \xi_2 = \frac{1}{2}(\beta(X)\xi_1 \wedge \beta(Y)\xi_2 + \beta(Y)\xi_1 \wedge \beta(X)\xi_2),$$

[3] Blaschke [3] showed that those surfaces that aren't determined are the Lie applicable surfaces. We shall explore this further in Section 5.1.

for any $X, Y \in \Gamma T\Sigma$ and $\xi_1, \xi_2 \in \Gamma f$. By Remark 2.5, the induced metric on the rank 2 bundle $f^\perp/f$ is non-degenerate. Therefore, the induced metric on the rank one bundle $\wedge^2(f^\perp/f)$ is definite and thus $\wedge^2(f^\perp/f)$ is a trivial bundle and we can identify c as a tensor in $S^2T^*\Sigma \otimes (\wedge^2 f)^*$. Now suppose that s is a curvature sphere of f with curvature subbundle T^s. Then $\beta(T^s)s = 0$ and since we may write any $\tau \in \Gamma(\wedge^2 f)$ as $\tau = \sigma \wedge \tilde{\sigma}$, for some $\sigma \in \Gamma s$ and $\tilde{\sigma} \in \Gamma f$, we have that

$$c(T^s, T^s)\tau = 0.$$

Hence, $c(T^s, T^s) = 0$. Therefore, at umbilic points $p \in \Sigma$ of f, $c_p = 0$ and away from umbilic points, for any $\tau \in \Gamma(\wedge^2 f)^\times$, $g := c\,\tau$ defines an indefinite metric on Σ whose null lines are the curvature subbundles T_1 and T_2. We shall refer to g as a representative metric of c and, since c is tensorial in $\wedge^2 f$, we have that any other representative metric of c is conformally equivalent to g. We shall thus refer to c as the conformal structure of f.

The product structure J induced by c acts as the id on T_1 and $-id$ on T_2, whereas the Hodge star operator $\star$ induced by c on $T^*\Sigma$ acts as the id on T_1^* and $-id$ on T_2^*. One can then deduce the following lemma:

Lemma 2.28. *$Q \in \Gamma End(T\Sigma)$ is trace-free and symmetric with respect to c if and only if*

$$\star Q = -J \circ Q.$$

Corollary 2.29. *Suppose that $Q \in \Gamma End(T\Sigma)$ is trace-free and symmetric with respect to c. Let g be a representative metric for c with induced Levi-Civita connection ∇. Then $d^\nabla \star Q = 0$, i.e., Q is divergence free, if and only if $d^\nabla Q = 0$.*

Proof. Since ∇ is the Levi-Civita connection for g, we have that $d^\nabla J = 0$. Then, using Lemma 2.28 and the Leibniz rule,

$$d^\nabla \star Q = -(d^\nabla J) \circ Q - J \circ d^\nabla Q = -J \circ d^\nabla Q,$$

and the result follows. $\square$

Lie-invariant metric

Now suppose that f is an umbilic-free Legendre map. Recall from Subsection 2.3.1 that the Lie cyclide splitting induces a skew-symmetric endomorphism $\mathcal{N} \in \Omega^1(S_1 \wedge S_2)$. By Lemma 2.20, $\mathcal{N} f \leq \Omega^1(f)$. Therefore, we may define a tensor $g^L \in \Gamma(S^2T^*\Sigma \otimes \mathrm{End}(\wedge^2 f))$ by

$$g^L(X,Y)\xi_1 \wedge \xi_2 = \frac{1}{2}(\mathcal{N}(X)\xi_1 \wedge \mathcal{N}(Y)\xi_2 + \mathcal{N}(Y)\xi_1 \wedge \mathcal{N}(X)\xi_2), \tag{2.2}$$

for any $X, Y \in \Gamma T\Sigma$ and $\xi_1, \xi_2 \in \Gamma f$. Since $\wedge^2 f$ has rank one, we identify g^L with a quadratic form. By Lemma 2.20, the curvature subbundles T_1 and T_2 are isotropic with respect to g^L and thus g^L is a representative metric of c.

Remark 2.30. *Unlike the conformal structure c, g^L may vanish at certain points. For example, by Corollary 2.22, if f is a Dupin cyclide then $g^L = 0$.*

Recall that given a space form $\mathfrak{Q}^3$ and space form projection $\mathfrak{f} : \Sigma \to \mathfrak{Q}^3$ of f with tangent plane congruence $\mathfrak{t} : \Sigma \to \mathfrak{P}^3$, we have that

$$\mathfrak{t} + \kappa_1 \mathfrak{f} \quad \text{and} \quad \mathfrak{t} + \kappa_2 \mathfrak{f}$$

are lifts of the curvature spheres s_1 and s_2, respectively. Now we may split the trivial connection $d = d_1 + d_2$, where d_i denotes the partial connection along T_i. Then one can then check that

$$\mathcal{N}(\mathfrak{t} + \kappa_1 \mathfrak{f}) = -\frac{d_1\kappa_1}{\kappa_1 - \kappa_2}(\mathfrak{t} + \kappa_2 \mathfrak{f}) \quad \text{and} \quad \mathcal{N}(\mathfrak{t} + \kappa_2 \mathfrak{f}) = \frac{d_2\kappa_2}{\kappa_1 - \kappa_2}(\mathfrak{t} + \kappa_1 \mathfrak{f}).$$

Hence,

$$g^L = (\kappa_1 - \kappa_2)^{-2} d_1\kappa_1 \odot d_2\kappa_2,$$

and thus g^L coincides with the Lie-invariant metric of [42, Theorem 1].

2.5.2 Darboux cubic form

Suppose that f is an umbilic-free Legendre map. For $X, Y, Z \in \Gamma T\Sigma$ and $\xi_1, \xi_2 \in \Gamma f$, define a map

$$\mathcal{C}(X, Y, Z)\xi_1 \wedge \xi_2 := (\mathcal{D}_X\mathcal{D}_Y\xi_1, \mathcal{N}_Z\xi_2) - (\mathcal{D}_X\mathcal{D}_Y\xi_2, \mathcal{N}_Z\xi_1).$$

Lemma 2.31. *$\mathcal{C}$ is a tensor taking values in $((T_1^*)^3 \oplus (T_2^*)^3) \otimes (\wedge^2 f)^*$.*

Proof. The tensorial nature of $\mathcal{C}$ follows from the fact that for any smooth function λ and for any $X, Y, Z \in \Gamma T\Sigma$ and $\xi \in \Gamma f$,

$$\mathcal{D}_X\mathcal{D}_Y(\lambda\xi) = \mathcal{D}_X\mathcal{D}_{\lambda Y}\xi = \mathcal{D}_{\lambda X}\mathcal{D}_Y\xi = \lambda\, \mathcal{D}_X\mathcal{D}_Y\xi \mod f^\perp$$

and by Lemma 2.20, $\mathcal{N}_Z f \leq f$.

Let $Z \in \Gamma T_1$ and $\sigma_1 \in \Gamma s_1$ and $\sigma_2 \in \Gamma s_2$. Then by Lemma 2.20, $\mathcal{N}_Z\sigma_2 = 0$, and thus for any $X, Y \in \Gamma T\Sigma$,

$$\mathcal{C}(X, Y, Z)\sigma_1 \wedge \sigma_2 = -(\mathcal{D}_X\mathcal{D}_Y\sigma_2, \mathcal{N}_Z\sigma_1).$$

If either of X or Y lies in T_2 then $\mathcal{D}_X\mathcal{D}_Y\sigma_2 \in \Gamma f^\perp$ and, since $\mathcal{N}_Z f \leq f$, this would imply that $\mathcal{C}(X, Y, Z) = 0$. A similar argument shows that if $Z \in \Gamma T_2$, then if X or Y lies in T_1 then $\mathcal{C}(X, Y, Z) = 0$. Hence,

$$\mathcal{C} \in \Gamma(((T_1^*)^3 \oplus (T_2^*)^3) \otimes (\wedge^2 f)^*).$$

□

We may thus write $\mathcal{C} = \mathcal{C}_1 + \mathcal{C}_2$ where $\mathcal{C}_i \in \Gamma((T_i)^3 \otimes (\wedge^2 f)^*)$. In terms of $\sigma_1 \in \Gamma s_1$ and $\sigma_2 \in \Gamma s_2$ we have that

$$\mathcal{C}_1(X, Y, Z)\sigma_1 \wedge \sigma_2 = -(\mathcal{D}_X\mathcal{D}_Y\sigma_2, \mathcal{N}_Z\sigma_1) \quad \text{and} \quad \mathcal{C}_2(X, Y, Z)\sigma_1 \wedge \sigma_2 = (\mathcal{D}_X\mathcal{D}_Y\sigma_1, \mathcal{N}_Z\sigma_2).$$

Remark 2.32. *By evaluating the Darboux cubic form $\mathcal{C}$ on $\tau := (\mathfrak{t} + \kappa_1 \mathfrak{f}) \wedge (\mathfrak{t} + \kappa_2 \mathfrak{f})$ one can check that*

$$\mathcal{C}\tau = (\kappa_2 - \kappa_1)(\kappa_{1,u} E\, du^3 + \kappa_{2,v} G\, dv^3),$$

in terms of curvature line coordinates (u, v). Hence, $\mathcal{C}\tau$ is in the same conformal class as the cubic form used in [42, Theorem 1].

It follows immediately from Corollary 2.21:

Lemma 2.33. *$\mathcal{C}_i$ vanishes if and only if the curvature sphere congruence s_i is constant along the leaves of T_i.*

We now recover a result stated in [42, Section 2]:

Corollary 2.34. *$\mathcal{C}$ vanishes if and only if f is a Dupin cyclide.*

2.6 Contact lift of $\mathbb{P}(\mathbb{R}^4)$

In this section we will recall (see for example [12, 41]) the contact lift of surfaces in $\mathbb{P}(\mathbb{R}^4)$ using the Klein correspondence between lines in $\mathbb{P}(\mathbb{R}^4)$ and points in the projective lightcone of $\mathbb{P}(\mathbb{R}^{3,3})$. Consider the six dimensional space $\wedge^2\mathbb{R}^4$. Let e_1, e_2, e_3, e_4 be the standard basis vectors of $\mathbb{R}^4$ and let

$$\omega := e_1 \wedge e_2 \wedge e_3 \wedge e_4 \in \wedge^2\mathbb{R}^4$$

be the volume element of $\mathbb{R}^4$. Then we define a scalar product $h(.,.)$ on $\wedge^2\mathbb{R}^4$ by

$$a \wedge b \wedge c \wedge d = h(a \wedge b, c \wedge d)\, e_1 \wedge e_2 \wedge e_3 \wedge e_4.$$

One can check that this scalar product has signature $(3, 3)$. Hence,

$$\wedge^2\mathbb{R}^4 \cong \mathbb{R}^{3,3}.$$

Furthermore, the lightcone of $\wedge^2\mathbb{R}^4$ is made up of decomposable elements of $\wedge^2\mathbb{R}^4$. We have a double covering $\mathrm{SL}(4) \to \mathrm{O}(3, 3)$ and thus an isomorphism $\phi : \mathfrak{sl}(4) \to \mathfrak{o}(3, 3)$ defined by

$$\phi(A)(v \wedge w) = (Av) \wedge w + v \wedge (Aw).$$

Suppose that Σ is a two dimensional manifold and let $F : \Sigma \to \mathbb{P}(\mathbb{R}^4)$ be an immersed surface, i.e., if we identify F as a rank one subbundle of $\underline{\mathbb{R}}^4$ then the derived bundle $F^{(1)}$ has constant rank 3. Let T_1, T_2 denote the (possibly complex conjugate) asymptotic directions of F, i.e. for any $X \in \Gamma T_1$, $Y \in \Gamma T_2$ and $\sigma \in \Gamma F$,

$$d_X d_X \sigma, d_Y d_Y \sigma \in \Gamma F^{(1)}.$$

We make the further assumption that

$$F^{(2)} := (F^{(1)})^{(1)} = \underline{\mathbb{R}}^4,$$

that is, for $X \in \Gamma T_1$, $Y \in \Gamma T_2$ and $\sigma \in \Gamma F$, $d_X d_Y \sigma$ never belongs to $F^{(1)}$. Now $f := F \wedge F^{(1)}$ is a rank 2 subbundle of $\wedge^2 \underline{\mathbb{R}}^4$ consisting of decomposable elements. Therefore, f takes values in $\mathcal{Z}$. Furthermore,

$$f^{(1)} = F^{(1)} \wedge F^{(1)} + F \wedge \underline{\mathbb{R}}^4$$

and

$$F \wedge F^{(1)} \wedge (F^{(1)} \wedge F^{(1)} + F \wedge \underline{\mathbb{R}}^4) = 0.$$

Thus, f satisfies the contact condition $f^{(1)} = f^{\perp}$. Furthermore, for asymptotic directions $X \in \Gamma T_1$, $Y \in \Gamma T_2$ and $\sigma \in \Gamma F$,

$$f = \langle \sigma \wedge d_X \sigma, \sigma \wedge d_Y \sigma \rangle.$$

Now

$$d_X(\sigma \wedge d_X \sigma), d_Y(\sigma \wedge d_Y \sigma) \in \Gamma f.$$

and

$$d_Y(\sigma \wedge d_X \sigma) = d_Y \sigma \wedge d_X \sigma + \sigma \wedge d_Y d_X \sigma$$

never belongs to f and, similarly, $d_X(\sigma \wedge d_Y \sigma)$ never belongs to f. Hence, f satisfies the immersion condition and f is a Legendre map with curvature spheres congruences $s_1 := \langle \sigma \wedge d_X \sigma \rangle$ and $s_2 := \langle \sigma \wedge d_Y \sigma \rangle$. In terms of projective differential geometry these represent the asymptotic lines of F.

Conversely, given a Legendre map $f : \Sigma \to \mathcal{Z}$, where $\mathcal{Z}$ is the space of lines in $\wedge^2 \mathbb{R}^4$, we have that

$$f = \langle \alpha \wedge \beta, \gamma \wedge \delta \rangle,$$

for some $\alpha, \beta, \gamma, \delta \in \Gamma \underline{\mathbb{R}}^4$. The condition that f takes values in $\mathcal{Z}$ implies that

$$\alpha \wedge \beta \wedge \gamma \wedge \delta = 0.$$

Therefore, $\delta \in \Gamma \langle \alpha, \beta, \gamma \rangle$ and $\gamma \wedge \delta = \gamma \wedge (\lambda \alpha + \mu \beta)$ for some smooth functions λ and μ. Hence, with $F := \lambda \alpha + \mu \beta$ and $W := \langle \gamma, \mu \alpha + \lambda \beta \rangle$ we have that

$$f = F \wedge W.$$

The contact condition $f^{(1)} = f^{\perp}$ then implies that $W \leq F^{(1)}$. Since $F \wedge W$ is a rank 2 subbundle of $\wedge^2 \underline{\mathbb{R}}^4$, we must have that $F^{(1)}$ has rank 3 and thus F is an immersion.

2.7 Ribaucour transformations

In [13], a modern treatment of Ribaucour transformations was developed in the realm of Lie sphere geometry. In this section we shall recall this construction and prove some results that will be useful to us later in Chapter 5.

Let $s,t \in \mathbb{N}$ with $s,t \geq 2$. Suppose that $f,\hat{f} : \Sigma \to \mathcal{Z}$ are Legendre maps in $\mathbb{R}^{s,t}$ and assume[4] that $\hat{f} \not\leq f^{\perp}$ and that f and $\hat{f}$ intersect at a point $s_0 := f \cap \hat{f}$. Then $s_0^{\perp}/s_0$ is a rank $s+t-2$ subbundle of $\underline{\mathbb{R}}^{s,t}/s_0$ and the induced metric on $s_0^{\perp}/s_0$ is non-degenerate with signature $(s-1,t-1)$. Let

$$\mathcal{N}_{f,\hat{f}} := (f+\hat{f})/s_0.$$

Then $\mathcal{N}_{f,\hat{f}}$ is a rank 2 subbundle of $s_0^{\perp}/s_0$ and the induced metric $\langle .,. \rangle$ on $\mathcal{N}_{f,\hat{f}}$ is non-degenerate with signature $(1,1)$. We then have a well-defined orthogonal projection $\pi : s_0^{\perp}/s_0 \to \mathcal{N}_{f,\hat{f}}$. From the contact condition on f and $\hat{f}$, one can quickly deduce the following lemma:

Lemma 2.35. $s_0^{(1)} \leq (f+\hat{f})^{\perp}$ *and* $(f+\hat{f})^{(1)} \leq s_0^{\perp}$.

We may now define a metric connection on $\mathcal{N}^{f,\hat{f}}$: for $\xi \in \Gamma(f+\hat{f})$,

$$\nabla^{f,\hat{f}}(\xi + s_0) = \pi(d\xi + s_0)$$

and make the following definition:

Definition 2.36. *If $\nabla^{f,\hat{f}}$ is flat then we say that s_0 is a Ribaucour sphere congruence and that f and $\hat{f}$ are Ribaucour transforms of each other.*

Now $f+\hat{f}$ is a rank 3 degenerate subbundle of $\underline{\mathbb{R}}^{s,t}$. If we let $l \leq f+\hat{f}$ be a rank 2 subbundle of $f+\hat{f}$ such that $l \cap s_0 = \{0\}$, then the induced metric on l has signature $(1,1)$. This yields a splitting

$$\underline{\mathbb{R}}^{s,t} = l \oplus l^{\perp}$$

and the trivial connection splits accordingly as

$$d = \mathcal{D}^{l} + \mathcal{D}^{l^{\perp}} + \mathcal{N}^{l,l^{\perp}},$$

where $\mathcal{D}^{l}$ is the induced connection on l, $\mathcal{D}^{l^{\perp}}$ is the induced connection on $l^{\perp}$ and

$$\mathcal{N}^{l,l^{\perp}} = d - (\mathcal{D}^{l} + \mathcal{D}^{l^{\perp}}) \in \Omega^1(Hom(l,l^{\perp}) \oplus Hom(l^{\perp},l)).$$

Proposition 2.37. *The vector bundle isomorphism*

$$\psi : l \to \mathcal{N}_{f,\hat{f}}, \quad \xi \mapsto \xi + s_0$$

preserves the metric and connection on l.

[4] In the case that $t = 2$, these assumptions are equivalent to assuming that f and $\hat{f}$ are pointwise distinct.

Proof. Suppose that $\xi_1, \xi_2 \in \Gamma l$. Then

$$\langle \psi(\xi_1), \psi(\xi_2) \rangle = \langle \xi_1 + s_0, \xi_2 + s_0 \rangle = (\xi_1, \xi_2).$$

Hence, the induced metric on l is isometric to $\langle ., . \rangle$ via ψ. Furthermore, for $\xi \in \Gamma l$,

$$\nabla^{f,\hat{f}}(\psi(\xi)) = \pi(d\xi + s_0) = \mathcal{D}^l \xi + s_0 = \psi(\mathcal{D}^l \xi).$$

Hence, ψ is connection preserving. $\square$

We then obtain an alternative characterisation of Ribaucour transforms:

Corollary 2.38. *f and $\hat{f}$ are Ribaucour transforms of each other if and only if the induced connection $\mathcal{D}^l$ is flat for some (and hence all) $l \leq f + \hat{f}$ of rank 2 such that $l \cap s_0 = \{0\}$.*

Remark 2.39. *Suppose that $l \cap s_0 = \{0\}$ and let $s := l \cap f$ and $\hat{s} := l \cap \hat{f}$. Then the condition that $\mathcal{D}^l$ be flat is equivalent to requiring s and $\hat{s}$ to be parallel subbundles of $\mathcal{D}^l$. In fact s being a parallel subbundle of $\mathcal{D}^l$ implies that $\hat{s}$ is parallel as well, and conversely.*

In the case that $t = 2$ it was shown in [13] that Definition 2.36 is equivalent to the classical definition of Ribaucour transform [3, 30, 31, 40, 68], that is, that the curvature directions of f and $\hat{f}$ correspond. Now assume that $(s,t) = (4,2)$ and let $s_1, s_2 \leq f$ denote the curvature sphere congruences of f and let $\hat{s}_1, \hat{s}_2 \leq \hat{f}$ denote the curvature sphere congruences of $\hat{f}$. Note that we are not necessarily assuming that f or $\hat{f}$ are umbilic-free, so the curvature spheres may coincide. Then we may assume that T_i is the curvature subbundle of s_i and $\hat{s}_i$ for $i \in \{1,2\}$. Let

$$l_i := s_i \oplus \hat{s}_i.$$

Then for any $\xi \in \Gamma l_i$ we have that $d\xi(T_i) \leq (f + \hat{f})$. Now let

$$s_\infty := l_1 \cap l_2.$$

Then for any $\sigma_\infty \in \Gamma s_\infty$, we have that $d\sigma_\infty(T_1) \leq f + \hat{f}$, since $\sigma_\infty \in \Gamma l_1$ and $d\sigma_\infty(T_2) \leq f + \hat{f}$, since $\sigma_\infty \in \Gamma l_2$. Therefore, as $T\Sigma = T_1 \oplus T_2$, $d\sigma_\infty \in \Omega^1(f + \hat{f})$. In fact s_∞ is the unique point in $\mathbb{P}(f + \hat{f})$ with the property that

$$s_\infty^{(1)} \leq f + \hat{f}$$

and this motivates the following definition:

Definition 2.40. *We call s_∞ the enveloping point of $f + \hat{f}$.*

Chapter 3

Deformations

It is well known that isothermic surfaces are the only surfaces in conformal geometry that admit non-trivial second order deformations [24]. In [56] it is shown that Ω- and Ω_0-surfaces are the surfaces in Lie geometry that admit non-trivial second order deformations. Motivated by this result we investigate deformations of smooth maps into projective space. We start by stating the definition of k-th order deformations of maps into homogeneous spaces [23, 22, 48, 51, 56].

Let N be a manifold on which a Lie group G acts smoothly and let $f, \hat{f} : \Sigma \to N$ be smooth maps.

Definition 3.1. *Let $k \in \mathbb{N} \cup \{0\}$. We say that f and $\hat{f}$ are k^{th}-order G-deformations of each other if there exists a smooth map $g : \Sigma \to G$ such that for all $p \in \Sigma$*

$$g^{-1}(p)\hat{f} \quad \textit{and} \quad f$$

agree to order k at p. We will call g a deforming transformation between f and $\hat{f}$. If there exists a constant deforming transformation between f and $\hat{f}$ we say that the deformation is trivial and that f and $\hat{f}$ are congruent. A map $f : \Sigma \to N$ is said to be G-deformable of order k if it admits a non-trivial k-th order G-deformation.

Remark 3.2. *We shall refer to "G-deformations" as just "deformations" when it has been made clear what the group G is.*

Remark 3.3. *k-th order contact at a point is transitive, i.e., if ϕ_1 and ϕ_2 agree to k-th order at a point p and ϕ_2 and ϕ_3 agree to k-th order at p, then ϕ_1 and ϕ_3 agree to k-th order at p.*

In this chapter we shall consider G-deformations of maps into projective space $\mathbb{P}(V)$ and submanifolds of projective space, where G is a Lie group acting on V with Lie algebra $\mathfrak{g}$. In doing so we obtain a result that characterises deformable maps by the existence of a certain $\mathfrak{g}$-valued one-form. We then study the uniqueness and triviality of deforming transformations. With these results in hand we set about proving well known results regarding applicability and rigidity in projective, conformal and Lie sphere geometry.

Remark 3.4. *In [51], Jensen considers deformations of maps into a homogeneous space $N = G/G_0$ using Cartan's method of moving frames. Given two maps $f, \hat{f} : \Sigma \to N$ and $k \in \mathbb{N}$, a system of exterior differential equations is derived on Σ which is satisfied if and only if the two maps are k-th order G-deformations of each other. By using his so called k-th order frames L_k associated to a map f, Jensen outlines methods for determining when k-th order G-deformations of f exist, finding all k-th order G-deformations of f and determining whether another map $\hat{f}$ is a k-th order G-deformation of f. Furthermore, Jensen shows that under certain generic assumptions on N there exists $k \geq 1$ such that k-th order deformability implies rigidity. These methods are then applied to the case of surfaces in $\mathbb{R}^3$ under the action of the equiaffine group, $SL(3, \mathbb{R}) \rtimes \mathbb{R}^3$.*

In order to recover the gauge-theoretic approach for Lie applicable surfaces (see [28]), we use a different approach for studying deformations of maps into projective spaces. In particular, we do not use Cartan's method of moving frames.

3.1 Deformations in projective space

Suppose that $N = \mathbb{P}(V)$ for some vector space V and suppose that G is a Lie group acting linearly on V. Let $\mathfrak{g}$ denote the Lie algebra of G.

Proposition 3.5. *$\phi, \hat{\phi} : \Sigma \to \mathbb{P}(V)$ agree to order k at $p \in \Sigma$ if and only if for any $v_0 \in V^*$, the sections $\sigma, \hat{\sigma}$ of ϕ and $\hat{\phi}$, respectively, such that*

$$v_0(\sigma) = v_0(\hat{\sigma}) = 1$$

agree to order k at p on the open set where they are defined.

Proof. ϕ and $\hat{\phi}$ agree to order k at p if and only if in any chart of $\mathbb{P}(V)$ they agree to order k at p. Let $v_0 \in V^*$ and $U := V \setminus \ker v_0$. Then $\mathbb{P}(U)$ is an open subset of $\mathbb{P}(V)$ and

$$\psi : \mathbb{P}(U) \to V, \quad [u] \mapsto u,$$

where $u \in [u]$ satisfies $v_0(u) = 1$, defines a chart $(\mathbb{P}(U), \psi)$ on $\mathbb{P}(V)$. Thus, ϕ and $\hat{\phi}$ agreeing to order k at p in this chart is equivalent to $\sigma := \psi(\phi)$ and $\hat{\sigma} := \psi(\hat{\phi})$ agreeing to order k at p. The result follows as the collection of charts defined by all $v_0 \in V^*$ is an atlas for $\mathbb{P}(V)$. $\square$

Let $j, k \in \mathbb{Z}$ and define $S_{j,k} := \{j, ..., k\}$ if $j \leq k$ and $S_{j,k} := \emptyset$ if $k < j$. Let W be a vector bundle over Σ, suppose that $X_j, ..., X_k \in T_p\Sigma$ and let $\sigma \in \Gamma W$. Then for $J \subset S_{j,k}$ with $J = \{j_1 < ... < j_l\}$ we let

$$d_{X_J}\sigma := d_{X_{j_1}}(d_{X_{j_2}}...(d_{X_{j_l}}\sigma)),$$

and

$$d_{X_\emptyset}\sigma := \sigma.$$

We will repeatedly use the Leibniz rule, i.e., if $\sigma, \xi \in \Gamma W$ and $J \subset S_{j,k}$, then

$$d_{X_J}(\sigma \otimes \xi) = \sum_{K \subset J} (d_{X_K}\sigma) \otimes (d_{X_{J \setminus K}}\xi).$$

Let $f : \Sigma \to \mathbb{P}(V)$ and $Y, X_1, ..., X_k \in \Gamma T\Sigma$. Then for $v_0 \in V^*$, $g : \Sigma \to G$ with $\theta = g^{-1}dg$ and $I, J \subset \{1, ..., k\}$, contemplate the following equation:

$$(d_{X_I}\theta(Y))d_{X_J}\sigma = \sum_{K \subset J} v_0((d_{X_I}\theta(Y))d_{X_K}\sigma)d_{X_{J \setminus K}}\sigma, \tag{3.1}$$

where $\sigma \in \Gamma f$ such that $v_0(\sigma) = 1$.

Assume that $k \in \mathbb{N}$. This section is mainly devoted to proving the following theorem:

Theorem 3.6. *$\hat{f} := gf$ is a k-th order deformation of f via g at $p \in \Sigma$ if and only if for all $Y, X_1, ..., X_{k-1} \in \Gamma T\Sigma$ and $v_0 \in V^*$, (3.1) holds for all $I, J \subset \{1, ..., k-1\}$ with $|I| + |J| < k - 1$ and for some $i \in \{0, ..., k-1\}$, (3.1) holds at p for all $I, J \subset \{1, ..., k-1\}$ with $|I| + |J| = k-1$ and $|I| = i$.*

We shall firstly examine the case in Theorem 3.6 where $i = 0$:

Lemma 3.7. *Suppose that f and $\hat{f} := gf$ are $(k-1)$-th order deformations of each other via g. Then f and $g^{-1}(p)gf$ agree to order k at $p \in \Sigma$ if and only if for any $v_0 \in V^*$ and $Y, X_1, ..., X_{k-1} \in \Gamma T\Sigma$,*

$$\theta(Y)d_{X_{S_{1,k-1}}}\sigma = \sum_{K \subset S_{1,k-1}} v_0(\theta(Y)d_{X_K}\sigma)d_{X_{S_{1,k-1} \setminus K}}\sigma,$$

at p, where $\sigma \in \Gamma f$ such that $v_0(\sigma) = 1$.

Proof. We shall use strong induction on k. Consider the case $k = 1$: f and $g^{-1}(p)gf$ agree to order 1 at p if and only if for any $v_0 \in V^*$, $v_0(g^{-1}(p)g\sigma)\sigma$ and $g^{-1}(p)g\sigma$ agree to order 1 at p where $\sigma \in \Gamma f$ such that $v_0(\sigma) = 1$. This holds if and only if for any $Y \in T_p\Sigma$,

$$g^{-1}(p)d_Y(g\sigma) = d_Y(v_0(g^{-1}(p)g\sigma)\sigma).$$

Now using the Leibniz rule and that $\theta_p(Y) = g^{-1}(p)d_Y g$, this holds if and only if

$$\theta_p(Y)\sigma + d_Y\sigma = v_0(\theta_p(Y)\sigma)\sigma + d_Y\sigma,$$

which is equivalent to

$$\theta_p(Y)d_\emptyset\sigma = \theta_p(Y)\sigma = v_0(\theta_p(Y)\sigma)\sigma = v_0(\theta_p(Y)d_\emptyset\sigma)d_\emptyset\sigma.$$

Hence, the proposition holds when $k = 1$.

Let $n \in \mathbb{N}$ and assume that the proposition holds for all $k < n$ and assume that f and $\hat{f}$ are $(n-1)$-th order deformations of each other. Let $Y, X_1, ..., X_{n-1} \in \Gamma T\Sigma$. Then for any

$K \subset \{1, ..., n-1\}$ with $|K| < n-1$ we have, by our inductive hypothesis,

$$\theta(Y) d_{X_K} \sigma = \sum_{L \subset K} v_0(\theta(Y) d_{X_L} \sigma) d_{X_{K \setminus L}} \sigma. \tag{3.2}$$

Since f and $\hat{f}$ are $(n-1)$-th order deformations of each other we have that for any $v_0 \in V^*$ and $X_1, ..., X_{n-1} \in \Gamma T \Sigma$,

$$g^{-1} d_{X_{S_{1,n-1}}} g\sigma - \sum_{K \subset S_{1,n-1}} v_0(g^{-1} d_{X_K} g\sigma) d_{X_{S_{1,n-1} \setminus K}} \sigma = 0,$$

where $\sigma \in \Gamma f$ such that $v_0(\sigma) = 1$. Differentiating at p with respect to $X_0 \in \Gamma T \Sigma$ we get, using the Leibniz rule and that $d_Y g^{-1} = -\theta(Y) g^{-1}$,

$$\begin{aligned}
0 &= -\theta_p(X_0) g^{-1}(p) d_{X_{S_{1,n-1}}} g\sigma + g^{-1}(p) d_{X_0} d_{X_{S_{1,n-1}}} g\sigma \\
&+ \sum_{K \subset S_{1,n-1}} (v_0(\theta_p(X_0) g^{-1}(p) d_{X_K} g\sigma) d_{X_{S_{1,n-1} \setminus K}} \sigma \\
&- v_0(g^{-1}(p) d_{X_0 X_K} g\sigma) d_{X_{S_{1,n-1} \setminus K}} \sigma - v_0(g^{-1}(p) d_{X_K} g\sigma) d_{X_0 X_{S_{1,n-1} \setminus K}} \sigma) \\
&= -\theta_p(X_0) g^{-1}(p) d_{X_{S_{1,n-1}}} g\sigma + \sum_{K \subset S_{1,n-1}} (v_0(\theta_p(X_0) g^{-1}(p) d_{X_K} g\sigma) d_{X_{S_{1,n-1} \setminus K}} \sigma \\
&+ d_{X_{S_{0,n-1}}} (g^{-1}(p) g\sigma) - d_{X_{S_{0,n-1}}} (v_0(g^{-1}(p) g\sigma)\sigma).
\end{aligned}$$

Thus, $v_0(g^{-1}(p) g\sigma)\sigma$ and $g^{-1}(p) g\sigma$ agree to order n at p if and only if

$$0 = -\theta_p(X_0) g^{-1}(p) d_{X_{S_{1,n-1}}} g\sigma + \sum_{K \subset S_{1,n-1}} v_0(\theta_p(X_0) g^{-1}(p) d_{X_K} g\sigma) d_{X_{S_{1,n-1} \setminus K}} \sigma. \tag{3.3}$$

Now, $v_0(g^{-1}(p) g\sigma)\sigma$ and $g^{-1}(p) g\sigma$ agree up to order $n-1$ at p, thus for any $K \subset S_{1,n-1}$,

$$g^{-1}(p) d_{X_K} g\sigma = d_{X_K}(v_0(g^{-1}(p) g\sigma)\sigma) = \sum_{L \subset K} v_0(g^{-1}(p) d_{X_L} g\sigma) d_{X_{K \setminus L}} \sigma.$$

Thus, (3.3) becomes

$$\begin{aligned}
0 &= -\theta_p(X_0) \sum_{K \subset S_{1,n-1}} v_0(g^{-1}(p) d_{X_K} g\sigma) d_{X_{S_{1,n-1} \setminus K}} \sigma \\
&+ \sum_{K \subset S_{1,n-1}} \sum_{L \subset K} v_0(\theta_p(X_0) v_0(g^{-1}(p) d_{X_L} g\sigma) d_{X_{K \setminus L}} \sigma) d_{X_{S_{1,n-1} \setminus K}} \sigma \\
&= -\sum_{K \subset S_{1,n-1}} v_0(g^{-1}(p) d_{X_K} g\sigma) \theta_p(X_0) d_{X_{S_{1,n-1} \setminus K}} \sigma \\
&+ \sum_{K \subset S_{1,n-1}} \sum_{L \subset K} v_0(g^{-1}(p) d_{X_L} g\sigma) v_0(\theta_p(X_0) d_{X_{K \setminus L}} \sigma) d_{X_{S_{1,n-1} \setminus K}} \sigma.
\end{aligned}$$

After relabelling we have that

$$\begin{aligned} 0 &= \sum_{K\subset S_{1,n-1}} v_0(g^{-1}(p)d_{X_K}g\sigma)(-\theta_p(X_0)d_{X_{S_{1,n-1}\setminus K}}\sigma \\ &+ \sum_{L\subset(S_{1,n-1}\setminus K)} v_0(\theta_p(X_0)d_{X_L}\sigma)d_{X_{(S_{1,n-1}\setminus K)\setminus L}}\sigma) \end{aligned}$$

vanishes. Using the inductive hypothesis (3.2) we then have

$$0 = -\theta_p(X_0)d_{X_{S_{1,n-1}}}\sigma + \sum_{K\subset S_{1,n-1}} v_0(\theta_p(X_0)d_{X_K}\sigma)d_{X_{S_{1,n-1}\setminus K}}\sigma.$$

Hence, the result holds for the case $k = n$. Hence, by induction the result is proved. □

Lemma 3.8. *Let $Y, X_1, ..., X_{k-1} \in \Gamma T\Sigma$ and suppose that for all $v_0 \in V^*$ and $I, J \subset \{1, ..., k-1\}$ with $|I| + |J| < k-1$, (3.1) holds. Then (3.1) holds at $p \in \Sigma$ for all $I, J \subset \{1, ..., k-1\}$ with $|I| = i \in \{0, , ..., k-1\}$ and $|I| + |J| = k-1$ if and only if (3.1) holds at p for all $I, J \subset \{1, ..., k-1\}$ with $|I| = i+1$ and $|I| + |J| = k-1$.*

Proof. Suppose that (3.1) holds at $p \in \Sigma$ for all $I, J \subset \{1, ..., k-1\}$ with $|I| = i \in \{0, , ..., k-1\}$ and $|I| + |J| = k-1$. Let $I, J \subset \{1, ..., k-1\}$ with $|I| = i+1$ and $|I| + |J| = k-1$. Without loss of generality, assume that $\min I < \min J$. Let a denote the smallest element of I and $\hat{I} := I\setminus\{a\}$. Then by our assumption

$$(d_{X_{\hat{I}}}\theta(Y))d_{X_J}\sigma = \sum_{K\subset J} v_0((d_{X_{\hat{I}}}\theta(Y))d_{X_K}\sigma)d_{X_{J\setminus K}}\sigma.$$

Differentiating this with respect to X_a at p and using the Leibniz rule we have that

$$\begin{aligned} & (d_{X_I}\theta(Y))d_{X_J}\sigma + (d_{X_{\hat{I}}}\theta(Y))d_{X_{\{a\}\cup J}}\sigma \\ = & \sum_{K\subset J}(v_0((d_{X_I}\theta(Y))d_{X_K}\sigma)d_{X_{J\setminus K}}\sigma + v_0((d_{X_{\hat{I}}}\theta(Y))d_{X_{\{a\}\cup K}}\sigma)d_{X_{J\setminus K}}\sigma \\ + & \sum_{K\subset J} v_0((d_{X_{\hat{I}}}\theta(Y))d_{X_K}\sigma)d_{X_{\{a\}\cup J\setminus K}}\sigma) \\ = & \sum_{K\subset J} v_0((d_{X_I}\theta(Y))d_{X_K}\sigma)d_{X_{J\setminus K}}\sigma + \sum_{L\subset\{a\}\cup J} v_0((d_{X_{\hat{I}}}\theta(Y))d_{X_L}\sigma)d_{X_{\{a\}\cup J\setminus L}}\sigma. \end{aligned}$$

By our supposition,

$$(d_{X_{\hat{I}}}\theta(Y))d_{X_{\{a\}\cup J}}\sigma = \sum_{L\subset\{a\}\cup J} v_0((d_{X_{\hat{I}}}\theta(Y))d_{X_L}\sigma)d_{X_{\{a\}\cup J\setminus L}}\sigma.$$

Thus,

$$(d_{X_I}\theta(Y))d_{X_J}\sigma = \sum_{K\subset J}(v_0((d_{X_I}\theta(Y))d_{X_K}\sigma)d_{X_{J\setminus K}}\sigma.$$

A similar argument can be used to prove the converse. □

Corollary 3.9. *Let $Y, X_1, ..., X_{k-1} \in \Gamma T\Sigma$ and suppose that for all $v_0 \in V^*$ and $I, J \subset \{1, ..., k-1\}$ with $|I| + |J| < k-1$, (3.1) holds. Then if (3.1) holds at $p \in \Sigma$ for all $I, J \subset \{1, ..., k-1\}$ with $|I| = i \in \{0, , ..., k-1\}$ and $|I| + |J| = k-1$, then (3.1) holds at p for all $I, J \subset \{1, ..., k-1\}$ with $|I| + |J| = k-1$.*

Proof of Theorem 3.6. Suppose that f and $g^{-1}(p)\hat{f}$ agree to order k at p. Then by Lemma 3.7, for any $v_0 \in V^*$ and $Y, X_1, ..., X_{k-1} \in \Gamma T\Sigma$, we have that (3.1) holds at p for all $I, J \subset \{1, ..., k-1\}$ with $|I| = 0$ and $|J| = k-1$. Then by Corollary 3.9, (3.1) holds at p for all $I, J \subset \{1, ..., k-1\}$ with $|I| + |J| = k-1$.

The converse follows by applying Corollary 3.9 and noting that in particular we have that for all $Y, X_1, ..., X_{k-1} \in \Gamma T\Sigma$ and $v_0 \in V^*$ (3.1) holds at p for the all $I, J \in \{1, ..., k-1\}$ with $|I| = 0$ and $|J| = k-1$. Then by Lemma 3.7, f and $g^{-1}(p)\hat{f}$ agree to order k at p.

□

Remark 3.10.
1. *It follows by Corollary 3.9 that we may replace "for some i" in Theorem 3.6 with "for all i".*
2. *When $i = k-1$, the statement (3.1) has the following cleaner form*

$$(d_{X_1 \ldots X_{k-1}} \theta(Y)) f(p) \leq f(p).$$

For practical application, however, it will be useful to consider different values of i.

From Theorem 3.6 we obtain the following corollaries:

Corollary 3.11. *f and $\hat{f} := gf$ are k-th order deformations of each other via $g : \Sigma \to G$ if and only if for all $Y, X_1, ..., X_{k-1} \in \Gamma T\Sigma$ and $v_0 \in V^*$, for some (and hence all) $i \in \{0, ..., k-1\}$, (3.1) holds for all $I, J \subset \{1, ..., k-1\}$ with $|I| = i$ and $|I| + |J| < k$.*

Corollary 3.12. *There exists k-th order deformations of f if and only if there exists $\eta \in \Omega^1(\mathfrak{g})$ such that η satisfies the Maurer-Cartan equation and for all $Y, X_1, ..., X_{k-1} \in \Gamma T\Sigma$ and $v_0 \in V^*$, for some (and hence all) $i \in \{0, ..., k-1\}$,*

$$(d_{X_I} \eta(Y)) d_{X_J} \sigma = \sum_{K \subset J} v_0((d_{X_I} \eta(Y)) d_{X_K} \sigma) d_{X_{J \setminus K}} \sigma, \tag{3.4}$$

holds for all $I, J \subset \{1, ..., k-1\}$ with $|I| = i$ and $|I| + |J| < k$, where $\sigma \in \Gamma f$ such that $v_0(\sigma) = 1$.

3.1.1 Submanifolds of projective space

In Sections 3.3 and 3.4 we study deformations of maps into quadrics. In view of this we shall consider maps into submanifolds of projective space. Suppose that S is a G-invariant submanifold of $\mathbb{P}(V)$.

Proposition 3.13. *Let $f : \Sigma \to S$ be a smooth map. Then f admits k-th order deformations (in S) if and only if there exists $\eta \in \Omega^1(\mathfrak{g})$ satisfying the Maurer-Cartan equation and for all*

$Y, X_1, ..., X_{k-1} \in \Gamma T\Sigma$ and $v_0 \in V^*$, for some (and hence all) $i \in \{0, ..., k-1\}$, (3.4) holds for all $I, J \in \{1, ..., k-1\}$ with $|I| = i$ and $|I| + |J| < k$.

Proof. This follows from the fact that k-th order contact of two maps in S is equivalent to k-th order contact as maps into $\mathbb{P}(V)$. □

3.1.2 Uniqueness of deforming transformations

We will now investigate the uniqueness of deforming transformations between two maps. Let $\underline{G} := \Sigma \times G$ denote the trivial bundle with fibre G over Σ. Let $G^{f,k}$ be the subset of $\underline{G}$ that at each point $p \in \Sigma$ is defined by

$$G_p^{f,k} := \{A \in G : A^{-1}f \text{ agrees to } k\text{-th order with } f \text{ at } p\}.$$

Notice that $G_p^{f,0}$ is the stabiliser of $f(p)$.

Lemma 3.14. *For all $p \in \Sigma$, $G_p^{f,k}$ is a subgroup of G.*

Proof. Clearly the identity is contained in $G_p^{f,k}$. Now suppose that $A, B \in G_p^{f,k}$. Then by Remark 3.3, $B^{-1}A^{-1}f$ agrees to k-th order with f. Thus, $AB \in G_p^{f,k}$. Furthermore, if $A \in G_p^{f,k}$, then since A is constant $f = A(A^{-1}f)$ agrees to k-th order with Af. Hence, $A^{-1} \in G_p^{f,k}$. □

Remark 3.15. *One should note that in general $G^{f,k}$ is not a subbundle of $\underline{G}$. For example if G does not act transitively on V, then the dimension of $G_p^{f,0}$ is not necessarily constant over Σ. If G does act transitively on V then $G^{f,0}$ is a subbundle of $\underline{G}$, however it is not clear whether this holds for $G^{f,k}$ with $k > 0$.*

Theorem 3.16. *Let $f, \hat{f} : \Sigma \to S$ be k-th order deformations of each other via a deforming transformation $g : \Sigma \to G$. Let $h \in \Gamma\underline{G}$. Then f and $\hat{f}$ are k-th order deformations via gh if and only if h is a k-th order deforming transformation between f and itself, i.e., for all $p \in \Sigma$, $h_p \in G_p^{f,k}$.*

Proof. Since f and $\hat{f}$ are k-th order deformations via g, we have that for each $p \in \Sigma$, $g^{-1}(p)\hat{f}$ agrees to k-th order with f at p. Since $h^{-1}(p)$ is constant, this is equivalent to $h^{-1}g^{-1}(p)\hat{f}$ agreeing to order k with $h^{-1}(p)f$ at p. It follows by Remark 3.3 that $h^{-1}(p)f$ agrees to order k with f at p if and only if $h^{-1}(p)g^{-1}(p)\hat{f}$ agrees to order k with f at p. □

Remark 3.17. *Since h is a k-th order deforming transformation between f and itself we must have that $hf = f$ and for all $Y, X_1, ..., X_{k-1} \in \Gamma T\Sigma$ and $v_0 \in V^*$, for some (and hence all) $i \in \{0, ..., k-1\}$,*

$$(d_{X_I}\theta_h(Y))d_{X_J}\sigma = \sum_{K \subset J} v_0((d_{X_I}\theta_h(Y))d_{X_K}\sigma)d_{X_{J\setminus K}}\sigma,$$

for all $I, J \subset \{1, ..., k-1\}$ with $|I| = i$ and $|I| + |J| < k$.

We will only be interested in deformations that are non-trivial. We thus have the following result:

Theorem 3.18. *Suppose that $f, \hat{f} : \Sigma \to S$ are k-th order deformations of each other via $g : \Sigma \to G$. Then this is a trivial deformation if and only if $g = Ah$ for some $A \in G$ and $h \in \Gamma G$ such that $h_p \in G_p^{f,k}$ for all $p \in \Sigma$.*

Proof. Suppose that $\hat{f}$ is congruent to f, i.e., there exists $A \in G$ such that $\hat{f} = Af$. Then A is a deforming transformation between f and $\hat{f}$. Let $h : \Sigma \to G$ such that $g = Ah$. Then by Theorem 3.16, for all $p \in \Sigma$, $h_p \in G_p^{f,k}$.

Conversely, if $g = Ah$ for some $A \in G$ and $h \in \Gamma G$ such that $h_p \in G_p^{f,k}$ for all $p \in \Sigma$, then by Theorem 3.16, f and $\hat{f}$ are k-th order deformations via $A = gh^{-1}$. In particular they are zeroth order deformations of each other via A. Hence, they are congruent. $\square$

Suppose now that $G^{f,k}$ is a subbundle of $\underline{G}$ and let $\mathfrak{g}^{f,k}$ denote the bundle of Lie algebras of $G^{f,k}$. Then Theorem 3.18 gives rise to the following corollary:

Corollary 3.19. *g induces a trivial deformation if and only if $\theta_g \in \Omega^1(\mathfrak{g}^{f,k})$.*

3.2 Projective 3-space

Fubini [44] and later Cartan [22] investigated projective applicability and rigidity of surfaces in projective 3-space. A modern account of this can be found in [41]. In this section we will use the results from Section 3.1 to study these notions.

3.2.1 Second order deformations

We will now consider projectively applicable surfaces, i.e., surfaces in $\mathbb{P}(\mathbb{R}^4)$, that admit non-trivial second order SL(4)-deformations. Let $F : \Sigma \to \mathbb{P}(\mathbb{R}^4)$ be a smooth map, where Σ is a two-dimensional manifold. By Proposition 3.13 with $i = 0$, F admits second order deformations if and only if there exists $\eta \in \Omega^1(\underline{\mathfrak{sl}(4)})$ satisfying the Maurer-Cartan equation and for all $v_0 \in (\mathbb{R}^4)^*$ and $X, Y \in \Gamma T\Sigma$

$$\eta\sigma = v_0(\eta\sigma)\sigma \tag{3.5}$$

and

$$\eta(X)d_Y\sigma = v_0(\eta(X)\sigma)d_Y\sigma + v_0(\eta(X)d_Y\sigma)\sigma, \tag{3.6}$$

where $\sigma \in \Gamma F$ such that $v_0(\sigma) = 1$.

Suppose that X and Y are linearly independent asymptotic directions of F, i.e., for any section σ of F,

$$d_{XX}\sigma, d_{YY}\sigma \in \Gamma F^{(1)},$$

where $F^{(1)}$ is the derived bundle of F, and assume that $d_{XY}\sigma$ lies nowhere in $F^{(1)}$. Suppose that F is projectively applicable. By equation (3.6) we have that

$$\eta(X)d_X\sigma = v_0(\eta(X)\sigma)d_X\sigma + v_0(\eta(X)d_X\sigma)\sigma.$$

Differentiating this in the Y direction gives

$$\begin{aligned}&(d_Y\eta(X))d_X\sigma + \eta(X)d_{YX}\sigma = \\ &d_Y(v_0(\eta(X)\sigma))d_X\sigma + v_0(\eta(X)\sigma)d_{YX}\sigma + d_Y(v_0(\eta(X)d_X\sigma))\sigma + v_0(\eta(X)d_X\sigma)d_Y\sigma.\end{aligned}$$

Since η satisfies the Maurer-Cartan equation the left hand side of this becomes

$$\begin{aligned}& (d_X\eta(Y))d_X\sigma - [\eta(Y), \eta(X)]d_X\sigma + \eta([Y, X])d_X\sigma + \eta(X)d_{YX}\sigma \\ = \;& d_X(\eta(Y)d_X\sigma) - \eta(Y)d_{XX}\sigma - [\eta(Y), \eta(X)]d_X\sigma + \eta([Y, X])d_X\sigma + \eta(X)d_{YX}\sigma \\ = \;& \eta(X)d_{YX}\sigma \, mod \, F^{(1)}.\end{aligned}$$

The right hand side is

$$v_0(\eta(X)\sigma)d_{YX}\sigma \, mod \, F^{(1)}.$$

Thus, modulo $F^{(1)}$,

$$\eta(X)d_{YX}\sigma = v_0(\eta(X)\sigma)d_{YX}\sigma \, mod \, F^{(1)}.$$

Similarly, one can show that

$$\eta(Y)d_{YX}\sigma = v_0(\eta(Y)\sigma)d_{YX}\sigma \, mod \, F^{(1)}.$$

Using that $\{\sigma, d_X\sigma, d_Y\sigma, d_{YX}\sigma\}$ forms a basis for $\mathbb{P}(\mathbb{R}^4)$ and that η takes values in $\mathfrak{sl}(4)$ and is thus trace free, we must have that $v_0(\eta\sigma) = 0$. Therefore,

$$\eta F = 0 \quad \text{and} \quad \eta F^{(1)} \leq \Omega^1(F).$$

Conversely if η satisfies

$$\eta F = 0 \quad \text{and} \quad \eta F^{(1)} \leq \Omega^1(F)$$

then clearly (3.5) and (3.6) hold and thus η yields a second order deformation of F.

Theorem 3.20. *$F : \Sigma \to \mathbb{P}(\mathbb{R}^4)$ admits second order deformations if and only if there exists $\eta \in \Omega^1(\underline{\mathfrak{sl}(4)})$ satisfying the Maurer-Cartan equation,*

$$\eta F = 0 \quad \textit{and} \quad \eta F^{(1)} \leq \Omega^1(F). \tag{3.7}$$

Lemma 3.21. *Suppose that $\eta \in \Omega^1(\underline{\mathfrak{sl}(4)})$ satisfies (3.7). Then η satisfies the Maurer-Cartan equation if and only if η is closed.*

Proof. First note that since η satisfies (3.7),

$$[\eta \wedge \eta]|_{F^{(1)}} = 0.$$

Thus, if η satisfies the Maurer-Cartan equation or η is closed we have that

$$(d\eta)F^{(1)} = 0.$$

Let X and Y be distinct asymptotic directions of F. Then for any $\sigma \in \Gamma F$,

$$\begin{aligned} 0 = d\eta(X,Y)\sigma &= (d_X\eta(Y) - d_Y\eta(X) - \eta([X,Y]))\sigma \\ &= d_X(\eta(Y)\sigma) - \eta(Y)d_X\sigma - d_Y(\eta(X)\sigma) + \eta(X)d_Y\sigma \\ &= -\eta(Y)d_X\sigma + \eta(X)d_Y\sigma, \end{aligned} \tag{3.8}$$

using the Leibniz rule and that $\eta F = 0$. Furthermore,

$$\begin{aligned} 0 &= d\eta(X,Y)d_X\sigma \\ &= (d_X\eta(Y) - d_Y\eta(X) - \eta([X,Y]))d_X\sigma \\ &= d_X(\eta(Y)d_X\sigma) - \eta(Y)d_{XX}\sigma - d_Y(\eta(X)d_X\sigma) + \eta(X)d_{YX}\sigma - \eta([X,Y])d_X\sigma. \end{aligned}$$

Therefore, as $d_{XX}\sigma \in \Gamma F^{(1)}$ we have that

$$0 = \eta(Y)d_X(\eta(Y)d_X\sigma) - \eta(Y)d_Y(\eta(X)d_X\sigma) + \eta(Y)\eta(X)d_{YX}\sigma.$$

Similarly, one can show that

$$0 = \eta(X)d_Y(\eta(X)d_Y\sigma) - \eta(X)d_X(\eta(Y)d_Y\sigma) + \eta(X)\eta(Y)d_{YX}\sigma.$$

Thus,

$$[\eta\wedge\eta](X,Y)d_{YX}\sigma = \eta(X)(d_X(\eta(Y)d_Y\sigma) - d_Y(\eta(X)d_Y\sigma)) + \eta(Y)(d_X(\eta(Y)d_X\sigma) - d_Y(\eta(X)d_X\sigma)).$$

Using (3.8), the right hand side becomes

$$\eta(X)(d_X(\eta(Y)d_Y\sigma) - d_Y(\eta(Y)d_X\sigma)) + \eta(Y)(d_X(\eta(X)d_Y\sigma) - d_Y(\eta(X)d_X\sigma)).$$

Using the Leibniz rule, one can then show that this is equivalent to

$$((d_X\eta(X))(d_Y\eta(Y)) - (d_Y\eta(X))(d_X\eta(Y)) + (d_X\eta(Y))(d_Y\eta(X)) - (d_Y\eta(Y))(d_X\eta(X)))\sigma.$$

By Proposition 3.13 with $i = 1$, for any $X_1, X_2 \in \Gamma T\Sigma$, $(d_{X_1}\eta(X_2))F \leq F$. This then implies that the right hand side vanishes. Thus, $[\eta \wedge \eta]d_{YX}\sigma = 0$. Then since $\underline{\mathbb{R}}^4 = F^{(1)} \oplus \langle d_{YX}\sigma\rangle$, $[\eta \wedge \eta] = 0$ and the result is proved. $\square$

Uniqueness and triviality of deforming transformations

We will now investigate the uniqueness of second order deforming transformations between two maps $F, \hat{F} : \Sigma \to \mathbb{P}(\mathbb{R}^4)$ and when a deforming transformation yields a trivial deformation.

By Theorem 3.16 the uniqueness of second order deforming transformations is determined up to the existence of $h \in \Gamma\mathrm{SL}(4)^{F,2}$. By Theorem 3.20, such a h satisfies

$$hF = F, \quad \theta_h F = 0 \quad \text{and} \quad \theta_h F^{(1)} \leq \Omega^1(F). \tag{3.9}$$

Now $hF = F$ implies that for any $\sigma \in \Gamma F$,

$$h\sigma = \lambda\sigma$$

for a smooth function λ. Thus, for any $X \in \Gamma T\Sigma$

$$(d_X h)\sigma + h d_X \sigma = \lambda d_X \sigma + (d_X \lambda)\sigma.$$

Using that $\theta_h F = 0$

$$h d_X \sigma = \lambda d_X \sigma + (d_X \lambda)\sigma.$$

Differentiating this condition with respect to $Y \in \Gamma T\Sigma$ we have that

$$h d_{YX} \sigma = \lambda d_{YX} \sigma + (d_Y \lambda) d_X \sigma + (d_X \lambda) d_Y \sigma + (d_{YX} \lambda)\sigma - (d_Y h) d_X \sigma.$$

Then, since h takes values in SL(4) and $\theta_h F^{(1)} \leq \Omega^1(F)$, we must have that $\lambda = \pm 1$. Furthermore,

$$h|_{F^{(1)}} = \pm id|_{F^{(1)}} \quad \text{and} \quad h|_{\underline{\mathbb{R}}^4/F} = \pm id|_{\underline{\mathbb{R}}^4/F}.$$

Thus, we may write

$$h = \pm(id + \xi),$$

where ξ satisfies $\xi|_{F^{(1)}} = 0$ and $im\xi \leq F$. Clearly ξ is trace-free, so $\xi \in \Gamma\underline{\mathfrak{sl}(4)}$. Hence, $h = \pm\exp(\xi)$.

One can easily check that if $h = \pm\exp(\xi)$, for some $\xi \in \Gamma\underline{\mathfrak{sl}(4)}$ satisfying $\xi|_{F^{(1)}} = 0$ and $im\xi \leq F$, then h satisfies (3.9) and thus $h \in \Gamma\mathrm{SL}(4)^{F,2}$.

Proposition 3.22. *Deforming transformations between maps $F, \hat{F} : \Sigma \to \mathbb{P}(\mathbb{R}^4)$ are determined up to right multiplication by $\pm\exp(\xi)$, for any $\xi \in \Gamma\underline{\mathfrak{sl}(4)}$ satisfying $\xi|_{F^{(1)}} = 0$ and $im\xi \leq F$.*

Using Theorem 3.18 and Corollary 3.19 we immediately obtain the following two corollaries:

Corollary 3.23. *Suppose that F and $\hat{F}$ are second order deformations of each other via $g : \Sigma \to SL(4)$. Then F and $\hat{F}$ are congruent if and only if $g = A\exp(\xi)$ for some $A \in SL(4)$ and $\xi \in \Gamma\underline{\mathfrak{sl}(4)}$ satisfying $\xi|_{F^{(1)}} = 0$ and $im\xi \leq F$.*

Corollary 3.24. *$g : \Sigma \to SL(4)$ yields a trivial second order deformation of F if and only if $g^{-1}dg = d\xi$, where $\xi \in \Gamma\underline{\mathfrak{sl}(4)}$ satisfying $\xi|_{F^{(1)}} = 0$ and $im\xi \leq F$.*

We have therefore proved the main theorem of this subsection:

Theorem 3.25. *F is projectively applicable if and only if there exists $\eta \in \Omega^1(\mathfrak{sl}(4))$, such that η is closed,*

$$\eta F = 0, \quad \eta F^{(1)} \leq \Omega^1(F)$$

and $\eta \neq d\xi$ for any $\xi \in \Gamma \underline{\mathfrak{sl}(4)}$ satisfying $\xi|_{F^{(1)}} = 0$ and $im\xi \leq F$.

Remark 3.26. *In Section 3.5 we shall see that the applicability of a map into $\mathbb{P}(\mathbb{R}^4)$ coincides with applicability of its contact lift. In this setting we shall see that the triviality of deformations can be identified by the vanishing of a certain two-tensor.*

3.2.2 Third order deformations

We shall now investigate what happens when two maps into projective 3-space are third order deformations of each other. Suppose that F and $\hat{F}$ are third order deformations of each other via $g : \Sigma \to \mathrm{SL}(4)$. Then by Theorem 3.25, $\theta = g^{-1}dg$ is closed and satisfies

$$\theta F = 0 \quad \text{and} \quad \theta F^{(1)} \leq \Omega^1(F).$$

Furthermore, by Proposition 3.13 with $i = 0$, for any $v_0 \in (\mathbb{R}^4)^*$ and $X, Y, Z \in \Gamma T\Sigma$,

$$\theta(X)d_{YZ}\sigma = v_0(\theta(X)d_{YZ}\sigma)\sigma + v_0(\theta(X)d_Y\sigma)d_Z\sigma + v_0(\theta(X)d_Z\sigma)d_Y\sigma + v_0(\theta(X)\sigma)d_{YZ}\sigma,$$

where $\sigma \in \Gamma f$ such that $v_0(\sigma) = 1$. Now suppose that Y is an asymptotic direction of F and $Z = Y$. Then $d_{YZ}\sigma \in \Gamma F^{(1)}$ and thus $\theta(X)d_{YZ}\sigma \in \Gamma F$. Hence, $v_0(\theta(X)d_Y\sigma) = 0$. Therefore, $\theta F^{(1)} = 0$. We will now use that θ is closed to show that $\theta = 0$: suppose that $X, Y, Z \in \Gamma T\Sigma$. Then, as θ is closed, we have that for any $\sigma \in \Gamma F$

$$d\theta(X, Y)d_Z\sigma = 0.$$

Since $\theta|_{F^{(1)}} = 0$, this is equivalent to

$$\theta(X)d_{YZ}\sigma - \theta(Y)d_{XZ}\sigma = 0.$$

Assume now that X and Y are distinct asymptotic directions of F. Then setting $Z = Y$ implies that $\theta(Y)d_{XY}\sigma = 0$, since $d_{YY}\sigma \in \Gamma F^{(1)}$. Similarly, setting $Z = X$ implies that $\theta(X)d_{YX}\sigma = 0$, which in turn implies that $\theta(X)d_{XY}\sigma = 0$. Therefore as $\{\sigma, d_X\sigma, d_Y\sigma, d_{XY}\sigma\}$ is a basis for $\underline{\mathbb{R}}^4$, $\theta = 0$. Thus we have proved the following theorem:

Theorem 3.27. *If F and $\hat{F}$ are third order deformations of each other then they are congruent.*

3.3 Hypersurfaces in the conformal n-sphere

In this section we will apply the results of Section 3.1 to examine deformations of hypersurfaces in conformal geometry. For a detailed analysis of conformal geometry see [6, 8]. Let $n \in \mathbb{N}$. Then we may view $\mathbb{S}^n$ as the projective light cone $\mathbb{P}(\mathcal{L})$ of $\mathbb{R}^{n+1,1}$, which is acted upon transitively by the orthogonal group $\mathrm{O}(n+1,1)$. Suppose that $f : \Sigma \to \mathbb{P}(\mathcal{L})$ is an immersion, where Σ is an $(n-1)$-dimensional manifold. We will view f as a null line subbundle of $\underline{\mathbb{R}}^{n+1,1}$. Note that as f is an immersion, the derived bundle $f^{(1)}$ of f is a codimension 1 subbundle of $f^{\perp}$. Let V be a sphere congruence enveloped by f, i.e., V is a bundle of $(n,1)$-planes such that $f^{(1)} \leq V$. Then let $\tilde{f}$ be a null-line subbundle of V complementary to f, i.e., $f \oplus \tilde{f}$ is a $(1,1)$-subbundle of V. Let $U := (f \oplus \tilde{f})^{\perp} \cap V$. Then $f^{(1)} = f \oplus U$ and $f^{\perp} = f \oplus U \oplus V^{\perp}$. We now have a splitting

$$\underline{\mathbb{R}}^{n+1,1} = f \oplus \tilde{f} \oplus U \oplus V^{\perp},$$

and thus a splitting of $\wedge^2 \underline{\mathbb{R}}^{n+1,1}$:

$$\wedge^2 \underline{\mathbb{R}}^{n+1,1} = f \wedge U \oplus f \wedge V^{\perp} \oplus U \wedge U \oplus U \wedge V^{\perp} \oplus f \wedge \tilde{f} \oplus \tilde{f} \wedge U \oplus \tilde{f} \wedge V^{\perp}.$$

3.3.1 Second order deformations

By Proposition 3.13 with $i = 0$, f admits second order deformations if and only if there exists $\eta \in \Omega^1(\underline{\mathfrak{o}(n+1,1)})$ satisfying the Maurer-Cartan equation, and for all $v_0 \in (\mathbb{R}^{n+1,1})^*$ and $X, Y \in \Gamma T\Sigma$

$$\eta\sigma = v_0(\eta\sigma)\sigma \quad \text{and} \quad \eta(X) d_Y \sigma = v_0(\eta(X)\sigma) d_Y \sigma + v_0(\eta(X) d_Y \sigma)\sigma, \tag{3.10}$$

where $\sigma \in \Gamma f$ such that $v_0(\sigma) = 1$. From the skew-symmetry of η it follows that $v_0(\eta\sigma) = 0$. Thus, (3.10) holds if and only if

$$\eta f = 0 \quad \text{and} \quad \eta f^{(1)} \leq \Omega^1(f),$$

or equivalently

$$\eta f = 0 \quad \text{and} \quad \eta U \leq \Omega^1(f).$$

This clearly holds if and only if

$$\eta \in \Omega^1(f \wedge U \oplus f \wedge V^{\perp}) = \Omega^1(f \wedge f^{\perp}).$$

Now $f \wedge f^{\perp}$ is an abelian subalgebra of $\underline{\mathfrak{o}(n+1,1)}$. Therefore, $[\eta \wedge \eta] = 0$ and the condition that η satisfies the Maurer-Cartan equation reduces to η being closed.

We shall now investigate the uniqueness and triviality of second order deformations.

Lemma 3.28. *Suppose f and $\hat{f}$ are second order deformations of each other via $g_1 : \Sigma \to O(n+1,1)$. If f and $\hat{f}$ are also second order deformations via $g_2 : \Sigma \to O(n+1,1)$ then*

$g_1 = \pm g_2$.

Proof. By Theorem 3.16, $h := g_1^{-1}g_2$ satisfies $hf = f$ and $\theta_h \in \Omega^1(f \wedge f^\perp)$. Thus, for any section $\sigma \in \Gamma f$, $h\sigma = \lambda\sigma$, for some smooth function λ. Differentiating this along $X \in \Gamma T\Sigma$ gives

$$(d_X h)\sigma + h d_X \sigma = (d_X \lambda)\sigma + \lambda d_X \sigma.$$

But since $\theta_h f = 0$, we have that

$$h d_X \sigma = (d_X \lambda)\sigma + \lambda d_X \sigma.$$

The orthogonality of h then gives that $\lambda = \pm 1$. Furthermore $h|_{f^{(1)}} = \pm id|_{f^{(1)}}$ and so for any $\nu \in \Gamma f^{(1)}$, $h\nu = \nu$. Differentiating this condition along $Y \in \Gamma T\Sigma$ gives

$$(d_Y h)\nu + h d_Y \nu = \pm d_Y \nu.$$

Then since $\theta_h f^\perp \leq f$, we have that $h|_{f^{(2)}} \equiv \pm id|_{f^{(2)}} \, mod\, f$. Now, $f^{(2)} := (f^{(1)})^{(1)} = \underline{\mathbb{R}}^{n+1,1}$, so we may write

$$h = \pm id + \xi,$$

where $\xi|_{f^{(1)}} = 0$ and $im\xi \leq f$. From the orthogonality of h one may deduce that ξ is skew-symmetric. Combined with $\xi|_{f^{(1)}} = 0$ and $im\xi \leq f$, this can only hold if $\xi = 0$, and the result is proved. □

Using Corollary 3.19 we get the following corollary:

Corollary 3.29. *Suppose f and $\hat{f}$ are second order deformations of each other via $g : \Sigma \to O(n+1,1)$. Then $\hat{f}$ is congruent to f if and only if g is constant and thus $\theta_g = 0$.*

We have thus arrived at the following theorem:

Theorem 3.30. *f admits non-trivial second order deformations if and only if there exists a closed non-zero one-form η taking values in $f \wedge f^\perp$.*

Remark 3.31. *In [5] it is shown that an η satisfying the conditions of Theorem 3.30 does not exist for $n > 3$.*

For the case $n = 3$ this characterises second order deformations as isothermic surfaces and in [64] it was proved that more can be said about where η takes values:

Proposition 3.32. *Suppose that $\eta \in \Omega^1(f \wedge f^\perp)$ is closed. Then $\eta \in \Omega^1(f \wedge f^{(1)})$.*

3.3.2 Third order deformations

We shall show that in $\mathbb{S}^3$ rigidity occurs at third order. Suppose that f and $\hat{f}$ are third order deformations of each other via $g : \Sigma \to \mathrm{O}(4,1)$. Then by Proposition 3.32, $\theta_g \in \Omega^1(f \wedge f^{(1)})$. Furthermore, by Proposition 3.13, for all $v_0 \in (\mathbb{R}^{4,1})^*$ and $X, Y, Z \in \Gamma T\Sigma$,

$$(d_X \theta_g(Y))\sigma = v_0((d_X \theta_g(Y))\sigma)\sigma$$

and

$$(d_X\theta_g(Y))d_Z\sigma = v_0((d_X\theta_g(Y))\sigma)d_Z\sigma + v_0((d_X\theta_g(Y))d_Z\sigma)\sigma,$$

where $\sigma \in \Gamma f$ such that $v_0(\sigma) = 1$. The skew-symmetry of $(d_X\theta_g(Y))$ implies that $v_0((d_X\theta_g(Y))\sigma) = 0$. Hence, $(d_X\theta_g(Y))\sigma = 0$. By the Leibniz rule this implies that $\theta_g(Y)d_X\sigma = 0$ and thus $\theta_g f^{(1)} = 0$. Therefore, $\theta_g = 0$. We have thus proved the following theorem:

Theorem 3.33. *If f and $\hat{f}$ are third order deformations of each other then they are congruent.*

3.4 Legendre maps

Let $s, t \in \mathbb{N}$ with $s, t \geq 2$ and let $n := s + t - 4$. Let $f : \Sigma \to \mathcal{Z}$ be a Legendre map, where Σ is a n-dimensional manifold and $\mathcal{Z}$ is the Grassmannian of isotropic two dimensional subspaces of $\mathbb{R}^{s,t}$. $\mathcal{Z}$ is acted upon transitively by $G := \mathrm{O}(s,t)$. Assume[1] that f has n curvature spheres $s_1, ..., s_n$ such that $s_i \cap s_j = \{0\}$ for all $i \neq j$ and let $T_1, ..., T_n$ denote the corresponding rank one curvature subbundles of $T\Sigma$. We identify f with the map $F : \Sigma \to Z$, defined by $F = \wedge^2 f$, where Z is the subset of $\mathbb{P}(\wedge^2\mathbb{R}^{s,t})$ defined by

$$Z := \{[v \wedge w] : v, w \in \mathcal{L} \text{ and } (v, w) = 0\}.$$

Z is acted upon smoothly and transitively by $O(s,t)$ via

$$A[v \wedge w] = [Av \wedge Aw].$$

Let $\tilde{f} : \Sigma \to \mathcal{Z}$ be complementary to f, i.e., $f \oplus \tilde{f}$ is a rank 4 bundle with signature $(2, 2)$. Let $U = (f \oplus \tilde{f})^\perp$. Then we have a splitting of $\underline{\mathbb{R}}^{s,t}$:

$$\underline{\mathbb{R}}^{s,t} = (f \oplus \tilde{f})^\perp \oplus_\perp U.$$

This induces a splitting of $\wedge^2\underline{\mathbb{R}}^{s,t}$:

$$\wedge^2\underline{\mathbb{R}}^{s,t} = \wedge^2 f \oplus f \wedge U \oplus f \wedge \tilde{f} \oplus \wedge^2 U \oplus \tilde{f} \wedge U \oplus \oplus \wedge^2 \tilde{f}.$$

3.4.1 Second order deformations

By Proposition 3.13 with $i = 0$ and $k = 1$ and $i = 1$ and $k = 2$, F admits second order deformations if and only if there exists $\eta \in \Omega^1(\underline{\mathfrak{o}(s,t)})$ satisfying the Maurer-Cartan equation and

$$\eta F \leq \Omega^1(F) \quad \text{and} \quad (d_X\eta(Y))F \leq F, \tag{3.11}$$

[1] With more effort one can prove that the results of this section hold without making this assumption.

for all $X, Y \in \Gamma T\Sigma$. Now $\eta F \leq \Omega^1(F)$ if and only if for linearly independent $\sigma, \xi \in \Gamma f$,

$$(\eta\sigma) \wedge \xi + \sigma \wedge (\eta\xi) = \eta(\sigma \wedge \xi) \in \Omega^1(F).$$

Since σ and ξ are linearly independent this is equivalent to

$$\eta f \leq \Omega^1(f).$$

Similarly, one can show that $(d_X\eta(Y))F \leq F$ is equivalent to $(d_X\eta(Y))f \leq f$. By the Leibniz rule, this holds if and only if for any section $\sigma \in \Gamma f$ and curvature directions $X_j \in \Gamma T_j$ and $X_k \in \Gamma T_k$,

$$d_{X_j}(\eta(X_k)\sigma) - \eta(X_k)d_{X_j}\sigma \in \Gamma f. \tag{3.12}$$

Now, as $\eta f \leq \Omega^1(f)$, $d_{X_j}(\eta(X_k)\sigma) \in \Gamma f_j$. Furthermore, $\eta(X_k)d_{X_j}\sigma$ is orthogonal to $d_{X_j}\sigma$. Therefore, as the metric on $\underline{\mathbb{R}}^{s,t}$ restricts to a non-degenerate metric on f_j/f, we can deduce that

$$d_{X_j}(\eta(X_k)\sigma), \eta(X_k)d_{X_j}\sigma \in \Gamma f.$$

Now, $d_{X_j}(\eta(X_k)\sigma) \in \Gamma f$ if and only if $\eta(X_k)\sigma \in \Gamma s_j$. As j and k were arbitrary and f is umbilic-free, this implies that $\eta f \equiv 0$. Also, $\eta(X_k)d_{X_j}\sigma \in \Gamma f$ for all j, k implies that $\eta f^{(1)} \leq \Omega^1(f)$. Thus, $\eta U \leq \Omega^1(f)$. Finally,

$$\eta f \equiv 0 \quad \text{and} \quad \eta U \in \Omega^1(f)$$

if and only if

$$\eta \in \Omega^1(\wedge^2 f \oplus f \wedge U) = \Omega^1(f \wedge f^\perp).$$

One can easily check that the converse is true, i.e., given $\eta \in \Omega^1(f \wedge f^\perp)$ satisfying the Maurer-Cartan equation, (3.11) holds.

Lemma 3.34. *Suppose that $\eta \in \Omega^1(f \wedge f^\perp)$. Then η satisfies the Maurer-Cartan equation if and only if it is closed. Furthermore, $\eta(T_i) \leq f \wedge f_i$ and $[\eta \wedge \eta] = 0$.*

Proof. Since $\eta f \equiv 0$, we have that

$$(d\eta + \frac{1}{2}[\eta \wedge \eta])f = d\eta f.$$

Let $X_i \in \Gamma T_i$ and $X_j \in \Gamma T_j$ for $i \neq j$ and $\sigma_i \in \Gamma s_i$. Then

$$\begin{aligned} d\eta(X_i, X_j)\sigma_i &= (d_{X_i}(\eta(X_j)) - d_{X_j}(\eta(X_i)) - \eta([X_i, X_j]))\sigma_i \\ &= d_{X_i}(\eta(X_j)\sigma_i) - \eta(X_j)d_{X_i}\sigma_i - d_{X_j}(\eta(X_i)\sigma_i) + \eta(X_i)d_{X_j}\sigma_i \\ &= -\eta(X_j)d_{X_i}\sigma_i + \eta(X_i)d_{X_j}\sigma_i, \end{aligned}$$

using again that $\eta f \equiv 0$. Since s_i is a curvature sphere, $d_{X_i}\sigma_i \in \Gamma f$ and thus $\eta(X_j)d_{X_i}\sigma_i = 0$. Therefore assuming that η satisfies the Maurer-Cartan equation or that it is closed implies that for all $i \neq j$, $X_i \in \Gamma T_i$ and $X_j \in \Gamma T_j$ and $\sigma_i \in \Gamma s_i$,

$$0 = \eta(X_i) d_{X_j} \sigma_i.$$

Thus, $\eta(X_i) \in \Gamma(f \wedge f_i)$ and

$$[\eta(X_i), \eta(X_j)] = 0.$$

Thus,

$$[\eta \wedge \eta](X_i, X_j) = 2[\eta(X_i), \eta(X_j)] = 0.$$

Therefore, since we may choose a basis of curvature directions $X_1, ..., X_n$ for $T\Sigma$, we have that $[\eta \wedge \eta] = 0$. Hence,

$$d\eta + \frac{1}{2}[\eta \wedge \eta] = d\eta$$

and the result follows. □

Thus, we have arrived at the following theorem:

Theorem 3.35. *f admits second order deformations if and only if there exists a closed one-form η taking values in $f \wedge f^{\perp}$.*

Uniqueness of second order deforming transformations

By Theorem 3.16, the uniqueness of second order deforming transformations of F is determined by the existence of $h \in \Gamma \mathrm{O}(s,t)^{F,2}$. By Theorem 3.35, $h \in \Gamma \mathrm{O}(s,t)^{F,2}$ if and only if

$$hF = F \quad \text{and} \quad \theta_h \in \Omega^1(f \wedge f^{\perp}).$$

Furthermore, $hF = F$ if and only if $hf = f$. Let $\sigma_i \in \Gamma s_i$ be a lift of one of the curvature spheres of f. Then, since $hf = f$ we have that

$$h\sigma_i = \nu,$$

for some $\nu \in \Gamma f$. Differentiating this condition with respect to the curvature direction X_i yields

$$(d_{X_i} h)\sigma_i + h d_{X_i} \sigma_i = d_{X_i} \nu.$$

Since $\theta_h \in \Omega^1(f \wedge f^{\perp})$, we have that $(d_{X_i} h)\sigma_i = 0$ and thus

$$h d_{X_i} \sigma_i = d_{X_i} \nu.$$

Since $d_{X_i}\sigma_i \in \Gamma f$ and $hf = f$, we must have that $d_{X_i}\nu \in \Gamma f$. Thus, $\nu \in \Gamma s_i$. Therefore, for some smooth function λ we have that $h\sigma_i = \lambda \sigma_i$. Differentiating this condition gives for all $X \in \Gamma T\Sigma$,

$$(d_X h)\sigma_i + h d_X \sigma_i = (d_X \lambda)\sigma_i + \lambda d_X \sigma_i. \tag{3.13}$$

Then the orthogonality of h and that $\theta_h f \equiv 0$ implies that $\lambda = \pm 1$. Therefore, $h|_{s_i} = \pm id|_{s_i}$. Now in the case that f has more than two curvature spheres it follows that $h|_f = \pm id|_f$. However in the case that $n = 2$, there are only two curvature spheres and there is another possibility to consider.

Lemma 3.36. *Suppose that f has exactly two curvature spheres s_1, s_2 with corresponding curvature subbundles T_1 and T_2, and that $h|_{s_1} = \pm id|_{s_1}$ and $h|_{s_2} = \mp id|_{s_2}$. Then f is a Dupin cyclide.*

Proof. Let $\sigma_1 \in \Gamma s_1$ and $\sigma_2 \in \Gamma s_2$ and let $X \in \Gamma T_1$ and $Y \in \Gamma T_2$. Then

$$d_X \sigma_1 = \alpha_1 \sigma_1 + \beta_1 \sigma_2 \quad \text{and} \quad d_Y \sigma_2 = \alpha_2 \sigma_1 + \beta_2 \sigma_2,$$

for smooth functions $\alpha_1, \alpha_2, \beta_1, \beta_2$. Now

$$\pm(\alpha_1 \sigma_1 + \beta_1 \sigma_2) = \pm d_X \sigma_1 = d_X(h\sigma_1) = (d_X h)\sigma_1 + h d_X \sigma_1 = \pm \alpha_1 \sigma_1 \mp \beta_1 \sigma_2,$$

since $\theta_h f \equiv 0$. Thus $\beta_1 = 0$. Similarly, one can show that $\alpha_2 = 0$. Then, since $X \in \Gamma T_1$ and $Y \in \Gamma T_2$ are arbitrary, f is a Dupin Cyclide (see Definition 2.17). $\square$

Remark 3.37. *In the case that f is a Dupin cyclide, there exists $\rho \in O(s,t)$ such that ρ restricts to the identity on S_1 and minus the identity on S_2, where S_1 and S_2 denote the Lie cyclides of f. If h satisfies $h|_{s_1} = \pm id|_{s_1}$ and $h|_{s_2} = \mp id|_{s_2}$, then $\tilde{h} := \rho h$ satisfies $\tilde{h} \in \Gamma O(s,t)^{F,2}$ and $\tilde{h}|_f = \pm id|_f$.*

Assume that $h|_f = \pm id|_f$. Then by (3.13), $h|_{f^{(1)}} = \pm id|_{f^{(1)}}$. By differentiating this condition again one finds that $h|_{f^{(2)}/f} = \pm id|_{f^{(2)}/f}$. Therefore we may write

$$h = \pm(id + \xi),$$

where ξ satisfies $\xi(\underline{\mathbb{R}}^{s,t}) \leq f$ and $\xi f^\perp \equiv 0$. Since $\xi(\underline{\mathbb{R}}^{s,t}) \leq f$, we have that $(\xi v, \xi w) = 0$ for all $v, w \in \Gamma \underline{\mathbb{R}}^{s,t}$. Now h is orthogonal so

$$(v,w) = (hv, hw) = (v + \xi v, w + \xi w) = (v,w) + (\xi v, w) + (v, \xi w).$$

Thus $(\xi v, w) + (v, \xi w) = 0$ and so $\xi \in \Gamma \underline{\mathfrak{o}(s,t)}$. Combining this with the fact that $\xi(\underline{\mathbb{R}}^{s,t}) \leq f$ and $\xi f^\perp \equiv 0$ gives that $\xi \in \Gamma \wedge^2 f$. Hence, $h = \pm \exp(\xi)$.

Conversely, if $h = \pm \exp(\xi)$, for some $\xi \in \Gamma(\wedge^2 f)$, then one can check that

$$hF = F, \quad \theta_h \in \Omega^1(f \wedge f^\perp).$$

Thus, $h \in \Gamma \mathrm{O}(s,t)^{F,2}$.

We have thus arrived at the following theorem:

Theorem 3.38. *Suppose that f and $\hat{f}$ are deforming transformations via g_1 and g_2. Then in* *the case that f is not a Dupin cyclide we have that $g_2 = \pm g_1 \exp(\xi)$ for some $\xi \in \Gamma(\wedge^2 f)$. In the case that f is a Dupin cyclide, either $g_2 = \pm g_1 \exp(\xi)$ or $g_2 = \pm \rho g_1 \exp(\xi)$.*

By Corollary 3.19, we have:

Corollary 3.39. *g is a trivial deformation if and only if $\theta_g = d\xi$ for some $\xi \in \Gamma(\wedge^2 f)$.*

Gauge orbit

Suppose that $\eta \in \Omega^1(f \wedge f^\perp)$ is closed. Notice that if $\xi \in \Gamma(\wedge^2 f)$, then $\eta + d\xi$ is closed and takes values in $\Omega^1(f \wedge f^\perp)$.

Definition 3.40. *We say that $\tilde{\eta} \in \Omega^1(f \wedge f^\perp)$ is gauge equivalent to η if there exists some $\xi \in \Gamma(\wedge^2 f)$ such that*

$$\tilde{\eta} = \eta + d\xi.$$

This defines an equivalence relation on closed one forms with values in $f \wedge f^\perp$ and we call the equivalence class of η the gauge orbit of η.

Remark 3.41. *We shall see in Proposition 3.48 that this definition coincides with the usual definition of gauge equivalence.*

By Corollary 3.39, we have the following theorem:

Theorem 3.42. *f admits non-trivial second order deformations if and only if there exists a closed one-form η taking values in $f \wedge f^\perp$ such that $[\eta] \neq [0]$.*

We will now introduce a tool that makes it easier to check when $[\eta] = [0]$. First note that for any $X, Y \in \Gamma T\Sigma$,

$$\sigma \mapsto \eta(X) d_Y \sigma$$

defines an endomorphism $f \to f$. Let q be the two-tensor defined by

$$q(X, Y) = \mathrm{tr}(\sigma \mapsto \eta(X) d_Y \sigma).$$

Lemma 3.43. *q is symmetric with $q(T_i, T_j) = 0$ for all $i \neq j$.*

Proof. By Lemma 3.34, for $X \in \Gamma T_i$, $\eta(X) \in \Gamma f \wedge f_i$. Therefore, for $i \neq j$, $\eta(X) f_j = 0$ and thus for any $\sigma \in \Gamma f$ and $Y \in \Gamma T_j$, $\eta(X) d_Y \sigma = 0$. Hence, $q(X, Y) = 0$. □

Lemma 3.44. *$q = 0$ if and only if $[\eta] = 0$.*

Proof. Suppose that $[\eta] = 0$. Then $\eta = d\xi$ for some $\xi \in \Gamma \wedge^2 f$. Now let $X, Y \in \Gamma T_i$, then one can check that $d_{XY}\sigma \in \Gamma s_i^\perp$ for any $\sigma \in \Gamma f$ and moreover, $d_{XY}\sigma \in \Gamma f^\perp$ if $\sigma \in \Gamma s_i$. Thus,

$$\eta(X) d_Y \sigma = (d_X \xi) d_Y \sigma = -\xi \, d_{XY} \sigma$$

lies in s_i and vanishes if $\sigma \in \Gamma s_i$. Hence, $q(X, Y) = 0$. By Lemma 3.43 this suffices to show that $q = 0$.

Conversely suppose that $q = 0$. Then we must have, $\eta(T_i) \leq s_i \wedge f_i$. Let σ_1 and σ_2 be lifts of the curvature spheres s_1 and s_2, respectively. Then for $i > 2$ let σ_i be the lift of the curvature sphere s_i such that $\sigma_i = \sigma_2 + \mu_i \sigma_1$, for some smooth functions μ_i. Thus, we may write

$$\eta = \sigma_1 \wedge d\sigma_2 \circ A_1 + \sigma_2 \wedge d\sigma_1 \circ A_2 + \sigma_3 \wedge d\sigma_1 \circ A_3 + ... + \sigma_r \wedge d\sigma_1 \circ A_r \, mod \, \Omega^1(f \wedge f),$$

where $A_i = \alpha_i id|_{T_i} \in \Gamma End(T_i)$ for some smooth functions α_i. Therefore, for $i \neq 1$, $X \in \Gamma T_1$ and $Y \in \Gamma T_i$,

$$\begin{aligned} 0 = d\eta(X, Y) &= d_X \sigma_i \wedge d\sigma_1(A_i(Y)) - d_Y \sigma_1 \wedge d\sigma_2(A_1(X)) \, mod \, \Omega^1(f \wedge \underline{\mathbb{R}}^{s,t}) \\ &= \alpha_i \, d_X \sigma_i \wedge d_Y \sigma_1 - \alpha_1 d_Y \sigma_1 \wedge d_X \sigma_2 \, mod \, \Omega^1(f \wedge \underline{\mathbb{R}}^{s,t}) \end{aligned}$$

Since $\sigma_i = \sigma_2 \, mod \, s_1$ we have that

$$0 = d\eta(X, Y) = \alpha_i \, d_X \sigma_2 \wedge d_Y \sigma_1 - \alpha_1 \, d_Y \sigma_1 \wedge d_X \sigma_2 \, mod \, \Omega^1(f \wedge \underline{\mathbb{R}}^{s,t}).$$

This implies that $\alpha_i = -\alpha_1$. Since i was arbitrary we have

$$\eta = \alpha_1(-\sigma_1 \wedge d\sigma_2|_{T_1} + \sigma_2 \wedge d\sigma_1|_{T_2} + \sigma_3 \wedge d\sigma_1|_{T_3} + ... + \sigma_r \wedge d\sigma_1|_{T_r}) \, mod \, \Omega^1(\wedge^2 f).$$

Now for $i > 2$, since $\sigma_i = \sigma_2 + \mu_i \sigma_1$,

$$\begin{aligned} \sigma_i \wedge d\sigma_1|_{T_i} &= (\sigma_2 + \mu_i \sigma_1) \wedge d\sigma_1|_{T_i} \, mod \, \Omega^1(\wedge^2 f) \\ &= (\sigma_2 \wedge d\sigma_1|_{T_i} + \sigma_1 \wedge d(\mu_i \sigma_1)|_{T_i}) \, mod \, \Omega^1(\wedge^2 f) \\ &= (\sigma_2 \wedge d\sigma_1|_{T_i} - \sigma_1 \wedge d\sigma_2|_{T_i}) \, mod \, \Omega^1(\wedge^2 f) \\ &= d(\sigma_2 \wedge \sigma_1)|_{T_i} \, mod \, \Omega^1(\wedge^2 f). \end{aligned}$$

Hence,

$$\eta = d(\alpha_1 \sigma_2 \wedge \sigma_1) \, mod \, \Omega^1(\wedge^2 f).$$

The closeness of η implies that

$$\eta = d(\alpha_1 \sigma_2 \wedge \sigma_1).$$

Thus, $[\eta] = 0$. □

We immediately obtain the following corollary:

Corollary 3.45. *f admits non-trivial second order deformations if and only if there exists a closed one-form η taking values in $f \wedge f^{\perp}$ such that $q \neq 0$.*

Corollary 3.46. *q is gauge invariant, i.e., if $\tilde{\eta} \in [\eta]$ then $\tilde{q} = q$.*

Proof. If $\tilde{\eta} \in [\eta]$ then $\tilde{\eta} - \eta = d\xi$ for some $\xi \in \Gamma f \wedge f$. Therefore, as $[d\xi] = [0]$,

$$\tilde{q} - q = 0$$

by Lemma 3.44. $\square$

Proposition 3.47. *η is closed if and only if $\{d^t := d + t\eta\}_{t\in\mathbb{R}}$ is a one-parameter family of flat metric connections.*

Proof. The curvature of the connection d^t is given by

$$R^{d^t} = td\eta + \frac{t^2}{2}[\eta \wedge \eta].$$

Thus, d^t is a one-parameter family of flat connections if only if η is closed and $[\eta \wedge \eta] = 0$. Then, by Lemma 3.34, if η is closed then $[\eta \wedge \eta] = 0$.

The fact that d^t is a metric connection follows by the skew-symmetry of η.

$\square$

Proposition 3.48. *Suppose that η is closed. Then for $\tilde{\eta} \in [\eta]$ with $\tilde{\eta} = \eta - d\tau$ for some $\tau \in \Gamma(\wedge^2 f)$ we have that*

$$d + t\tilde{\eta} = \exp(t\tau) \cdot (d + t\eta).$$

Proof. Since, $\tau \in \Gamma f \wedge f$, we have that $\tau f^{\perp} = 0$. Thus, for $v \in \Gamma\underline{\mathbb{R}}^{s,t}$,

$$\begin{aligned}
\exp(t\tau) \cdot (d + t\eta)v &= \exp(t\tau)(d + t\eta)\exp(-t\tau)v \\
&= \exp(t\tau)(dv - td(\tau v) + t\eta v) \\
&= dv - td(\tau v) + t\eta v + t\tau dv \\
&= (d + t(-d\tau + \eta))v \\
&= (d + t\tilde{\eta})v,
\end{aligned}$$

and the result follows. $\square$

3.4.2 Third order deformations

In this subsection we shall show that rigidity occurs at third order for Legendre maps. Suppose that F and $\hat{F}$ are third order deformations of each other via $g : \Sigma \to \mathrm{O}(s,t)$. Then by Theorem 3.35, $\theta = g^{-1}dg \in \Omega^1(f \wedge f^{\perp})$ and θ is closed. Now by Proposition 3.13 with $i = 2$, for $j, k, l \in \{1, ..., n\}$ and curvature directions $X_j \in \Gamma T_j$, $X_k \in \Gamma T_k$ and $X_l \in \Gamma T_l$,

$$(d_{X_j X_k}\theta(X_l))F \leq F.$$

Thus, for any linearly independent section $\sigma, \xi \in \Gamma f$,

$$((d_{X_j X_k}\theta(X_l))\sigma) \wedge \xi + \sigma \wedge ((d_{X_j X_k}\theta(X_l))\xi) = (d_{X_j X_k}\theta(X_l))(\sigma \wedge \xi) \in \Gamma F.$$

Therefore, as σ and ξ are linearly independent, this is equivalent to

$$(d_{X_j X_k}\theta(X_l))f \leq f. \tag{3.14}$$

Let $\sigma \in \Gamma f$. Then by the Leibniz rule, equation (3.14) implies that

$$d_{X_j}((d_{X_k}\theta(X_l))\sigma) - (d_{X_k}\theta(X_l))d_{X_j}\sigma \in \Gamma f. \tag{3.15}$$

Now, $(d_{X_k}\theta(X_l))\sigma \in \Gamma f$, thus $d_{X_j}((d_{X_k}\theta(X_l))\sigma) \in \Gamma f_j$. Furthermore, as $(d_{X_k}\theta(X_l))$ is skew-symmetric, $(d_{X_k}\theta(X_l))d_{X_j}\sigma$ is orthogonal to $d_{X_j}\sigma$. Thus equation (3.15) holds if and only if

$$d_{X_j}((d_{X_k}\theta(X_l))\sigma) \in \Gamma f \quad \text{and} \quad (d_{X_k}\theta(X_l))d_{X_j}\sigma \in \Gamma f.$$

Now $d_{X_j}((d_{X_k}\theta(X_l))\sigma) \in \Gamma f$ implies that

$$(d_{X_k}\theta(X_l))\sigma \in \Gamma s_j.$$

Since j was arbitrary, we have that $(d_{X_k}\theta(X_l))\sigma = 0$ for all k, l. By the Leibniz rule this implies that

$$d_{X_k}(\theta(X_l)\sigma) - \theta(X_l)d_{X_k}\sigma = 0,$$

and since $\theta(X_l)f = 0$, we have for all k, l,

$$\theta(X_l)d_{X_k}\sigma = 0.$$

Hence, $\theta f^{\perp} \equiv 0$ and thus $\theta \in \Omega^1(\wedge^2 f)$. One can then check that θ being closed implies that $\theta \equiv 0$ and thus g is constant.

Theorem 3.49. *If f and $\hat{f}$ are third order deformations of each other then they are congruent.*

3.5 Projective applicability revisited

In Section 2.6 we showed how surfaces in $\mathbb{P}(\mathbb{R}^4)$ could be represented by their contact lifts in $\mathbb{R}^{3,3}$. In this section we shall prove a result known to Fubini, [44], that the applicability of surfaces in $\mathbb{P}(\mathbb{R}^4)$ is equivalent to the applicability of their contact lifts.

Recall that the contact lift of a surface $F : \Sigma \to \mathbb{P}(\mathbb{R}^4)$ is given by

$$f = F \wedge F^{(1)},$$

and the derived bundle of this contact lift is

$$f^{(1)} = F^{(1)} \wedge F^{(1)} + F \wedge \underline{\mathbb{R}}^4.$$

Recall also that we have an isomorphism $\phi : \mathfrak{sl}(4) \to \mathfrak{o}(3,3)$, defined by

$$\phi(A)\,(v \wedge w) = Av \wedge w + v \wedge Aw.$$

Since ϕ is constant, ϕ intertwines the trivial connections on $\underline{\mathfrak{sl}(4)}$ and $\underline{\mathfrak{o}(3,3)}$. Let $\Theta \leq \underline{\mathfrak{sl}(4)}$

denote the subbundle of $\underline{\mathfrak{sl}(4)}$ such that $A \in \Gamma\Theta$ if and only if

$$AF = 0 \quad \text{and} \quad AF^{(1)} \leq F.$$

Then $A \in \Gamma\Theta$ implies that

$$\phi(A)f = (A\,F) \wedge F^{(1)} + F \wedge (A\,F^{(1)}) = 0.$$

Furthermore,

$$\phi(A)f^{(1)} = (A\,F^{(1)}) \wedge F^{(1)} + F^{(1)} \wedge (A\,F^{(1)}) + (A\,F) \wedge \underline{\mathbb{R}}^4 + F \wedge (A\,\underline{\mathbb{R}}^4) \leq F \wedge F^{(1)} = f.$$

Therefore, $\phi(A) \in \Gamma(f \wedge f^{\perp})$. Now ϕ injects and Θ has the same rank as $f \wedge f^{\perp}$. Thus, $\Theta \cong f \wedge f^{\perp}$ via ϕ. Hence, via ϕ, closed one forms taking values in Θ are in one-to-one correspondence with closed one forms taking values in $f \wedge f^{\perp}$. It remains to check that the triviality of these one-forms is preserved by ϕ. Let Ψ denote the subbundle of Θ defined by $A \in \Gamma\Psi$ if and only if

$$AF^{(1)} = 0 \quad \text{and} \quad imA \leq F.$$

Then as $\Psi \leq \Theta$, $A \in \Gamma\Psi$ implies that $\phi(A)f = 0$. Furthermore, it is easy to check that $\phi(A)f^{(1)} = 0$. Hence, $\phi(A) \in \Gamma(\wedge^2 f)$. Now Ψ and $\wedge^2 f$ both have rank 1. Thus, $\Psi \cong \wedge^2 f$ via ϕ. Therefore, as ϕ preserves the connection on $\underline{\mathfrak{sl}(4)}$, we have that a one-form $\omega \in \Omega^1(\underline{\mathfrak{sl}(4)})$ is of the form $\omega = d\xi$ for some $\xi \in \Gamma\Psi$ if and only if $\phi(\omega) = d\zeta$ for some $\zeta \in \Gamma(\wedge^2 f)$. We have thus arrived at the following theorem:

Theorem 3.50. *A surface in projective* 3*-space is applicable if and only if its contact lift is applicable.*

Further work

The examples considered in this chapter are examples of R-spaces [71]. Since any R-space can be represented as a submanifold of projective space, we may apply Theorem 3.6 to consider deformations in these more general spaces. It would be interesting to see how certain properties of R-spaces affect the deformations of these spaces. For example, how does the height of an R-space affect the rigidity of submanifolds in this space.

Chapter 4

Ω- and Ω_0-surfaces

In this chapter we shall restrict our attention to Legendre maps in $\mathbb{R}^{4,2}$. To deal with the non-uniqueness posed by gauge transformations, we introduce the middle connection which we shall use extensively in the following chapters. We then show that applicable surfaces in this context coincide with the Ω- and Ω_0-surfaces of Demoulin, [35, 36, 34] and then recover the isothermic sphere congruences that these surfaces envelope. In the final section of this chapter we recover the associate surface of an Ω-surface in Euclidean 3-space that was discovered by Demoulin [34] and show that a system of O-surfaces [65] arises from this construction.

Remark 4.1. *It should be noted that it is possible for Legendre maps to be Lie applicable in more than one way, i.e., admit more than one gauge-orbit of closed one-forms. The case that a Legendre map is Lie applicable in three parameters worth of ways has been studied in [43, 56].*

Let $\mathcal{Z}$ be the space of lines in $\mathbb{R}^{4,2}$. Let $f : \Sigma \to \mathcal{Z}$ be a Legendre map, with two distinct curvature spheres s_1 and s_2 with respective curvature bundles $T_1, T_2 \leq T\Sigma$. By Corollary 3.45, f is Lie applicable, i.e., admits non-trivial second order deformations, if and only if there exists a closed $\eta \in \Omega^1(f \wedge f^\perp)$ such that the quadratic differential $q \in \Gamma S^2(T\Sigma)$ defined by

$$q(X, Y) = tr(\sigma \mapsto \eta(X) d_Y \sigma)$$

is non-zero. Clearly, Lie applicability is a Lie invariant notion and we have the following proposition:

Proposition 4.2. *Suppose that f is Lie applicable with associated closed one form η. Then for any $A \in O(4,2)$, $\eta^A := Ad(A) \cdot \eta$ is a closed one-form which takes values in $f^A \wedge (f^A)^\perp$, where $f^A := Af$. Furthermore, the quadratic differential of η^A coincides with that of η.*

Proof. The closure of η^A follows trivially since A is constant and since $A \in \mathrm{O}(4,2)$ we have that

$$Ad(A) \cdot (f \wedge f^\perp) = Af \wedge Af^\perp = f^A \wedge (f^A)^\perp.$$

Hence, $\eta^A \in \Omega^1(f^A \wedge (f^A)^\perp)$. Now if $\sigma \in \Gamma f$, then

$$\eta^A(X)d_Y(A\sigma) = (Ad(A)\cdot\eta(X))Ad_Y\sigma = A\eta(X)d_Y\sigma.$$

Therefore the quadratic differential of η^A,

$$q^A(X,Y) = tr(A\sigma \mapsto \eta^A(X)d_Y(A\sigma))$$

coincides with q. $\square$

4.1 The Middle Connection

Suppose that f is Lie applicable with associated closed one-form η. In this section we shall see that within the gauge orbit of η there exists a distinguished gauge potential η^m which we shall call the middle connection.

Recall from Subsection 2.3.1 that we may split $\underline{\mathbb{R}}^{4,2}$ using the Lie cyclides

$$\underline{\mathbb{R}}^{4,2} = S_1 \oplus_\perp S_2.$$

This then induces the splitting $\underline{\mathfrak{o}(4,2)} = \mathfrak{h} + \mathfrak{m}$, where

$$\mathfrak{h} := S_1 \wedge S_1 \oplus S_2 \wedge S_2 \quad \text{and} \quad \mathfrak{m} := S_1 \wedge S_2.$$

Suppose that f is Lie applicable with non-trivial closed one-form $\eta \in \Omega^1(f \wedge f^\perp)$. We may split η into $\eta_\mathfrak{h}$ and $\eta_\mathfrak{m}$ where $\eta_\mathfrak{h} \in \Omega^1(\mathfrak{h})$ and $\eta_\mathfrak{m} \in \Omega^1(\mathfrak{m})$. We will now show that within the gauge-orbit of η there exists a unique gauge potential such that $\eta_\mathfrak{m} \in \Omega^1(\wedge^2 f)$.

Proposition 4.3. *$\eta_\mathfrak{h}$ is invariant of gauge transformation.*

Proof. Firstly, notice that $\wedge^2 f \leq \mathfrak{m}$, since for any $\xi \in \Gamma(\wedge^2 f)$ there exists $\sigma_1 \in \Gamma s_1$ and $\sigma_2 \in \Gamma s_2$ such that $\xi = \sigma_1 \wedge \sigma_2$. Now for any $X \in \Gamma T_1$,

$$d_X\xi = (d_X\sigma_1)\wedge\sigma_2 + \sigma_1 \wedge d_X\sigma_2 = \sigma_1 \wedge d_X\sigma_2 \, mod \wedge^2 f,$$

since $d_X\sigma_1 \in \Gamma f$. Thus, $d_X\xi \in \Gamma(S_1 \wedge S_2)$. Similarly, one can show that $d_Y\xi \in \Gamma(S_1 \wedge S_2)$, for any $Y \in \Gamma T_2$. Hence, $d\xi \in \Omega^1(S_1 \wedge S_2)$. $\square$

Proposition 4.4. *Modulo $\Omega^1(\wedge^2 f)$, $\eta_\mathfrak{m} = d\xi$ for some $\xi \in \Gamma(\wedge^2 f)$.*

Proof. Let $\sigma_1 \in \Gamma s_1$ and $\sigma_2 \in \Gamma s_2$ be lifts of the curvature spheres. Then we may write

$$\eta = (\alpha_1\,\sigma_1 \wedge d\sigma_1 + \alpha_2\,\sigma_2 \wedge d\sigma_2 + \beta_1\,\sigma_1 \wedge d\sigma_2 + \beta_2\,\sigma_2 \wedge d\sigma_1)\, mod\, \Omega^1(\wedge^2 f),$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2$ are smooth functions. In this case

$$\eta_\mathfrak{m} = (\beta_1\,\sigma_1 \wedge d\sigma_2 + \beta_2\,\sigma_2 \wedge d\sigma_1)\, mod\, \Omega^1(\wedge^2 f),$$

for some one form ω. Now $d\sigma_1 \curlywedge d\sigma_1, d\sigma_2 \curlywedge d\sigma_2 \in \Omega^2(f \wedge f^\perp)$ and thus

$$0 = d\eta = \beta_1 \, d\sigma_1 \curlywedge d\sigma_2 + \beta_2 \, d\sigma_2 \curlywedge d\sigma_1 \, mod \, \Omega^2(f \wedge f^\perp).$$

Therefore $\beta_2 = -\beta_1$ and

$$\eta_{\mathfrak{m}} = d(\beta_1 \sigma_1 \wedge \sigma_2) \, mod \, \Omega^1(\wedge^2 f),$$

and the result is proved. □

Corollary 4.5. *There exists a unique gauge potential of $[\eta]$ with $\eta_{\mathfrak{m}} \in \Omega^1(\wedge^2 f)$.*

Following Corollary 4.5 we have the following definition:

Definition 4.6. *We call the gauge potential in $[\eta]$ with $\eta_{\mathfrak{m}} \in \Omega^1(\wedge^2 f)$ the middle connection and denote it η^m.*

Therefore, since q is gauge invariant by Corollary 3.46 and $(\wedge^2 f) f^\perp = 0$, we can deduce the following corollary:

Corollary 4.7. *For any $X, Y \in \Gamma T\Sigma$,*

$$q(X, Y) = tr(\sigma \mapsto \eta_{\mathfrak{h}}(X) d_Y \sigma).$$

4.2 Ω- and Ω_0-surfaces

In this section we shall define Ω- and Ω_0-surfaces and see that this definition coincides with the classical definition of Demoulin, [35, 36, 34].

Definition 4.8. *Suppose that $\eta \in \Omega^1(f \wedge f^\perp)$ is closed with non-zero quadratic differential q. We say that f is an Ω-surface (Ω_0-surface) if q is non-degenerate (degenerate).*

Remark 4.9. *Notice that in this definition we do not exclude umbilic points. For the majority of our analysis and in this chapter in particular we will avoid such points.*

Assume that f is umbilic-free, i.e., the curvature spheres s_1 and s_2 of f never coincide. In this case T_1 and T_2 are rank one subbundles of $T\Sigma$. We shall split the connection d on $\underline{\mathbb{R}}^{4,2}$ into the partial connections d_1 and d_2, where d_i denotes differentiation along T_i. Suppose that q is a quadratic differential with respect to the conformal structure $[c]$. Up to rescaling q by ± 1 and reordering s_1 and s_2 we may assume that

$$q = -\epsilon^2 q_1 + q_2,$$

where $\epsilon \in \{0, 1, i\}$, and $q_1 \in \Gamma(T_1^*)^2$ and $q_2 \in \Gamma(T_2^*)^2$ are positive definite quadratic forms. Then q_1 and q_2 determine unique lifts $\sigma_1 \in \Gamma s_1$ and $\sigma_2 \in \Gamma s_2$ (up to sign) such that

$$q_1 = (d\sigma_2, d\sigma_2) \quad \text{and} \quad q_2 = (d\sigma_1, d\sigma_1).$$

Thus, q determines a unique one form $\eta_{\mathfrak{h}} \in \Omega^1(\mathfrak{h} \cap (f \wedge f^\perp))$ such that

$$q(X,Y) = \mathrm{tr}(\sigma \mapsto \eta_{\mathfrak{h}}(X) d_Y \sigma),$$

namely,

$$\eta_{\mathfrak{h}} = -\sigma_1 \wedge d_2\sigma_1 + \epsilon^2 \sigma_2 \wedge d_1\sigma_2.$$

Let $\omega := \omega_1 + \omega_2$ be a one form with $\omega_1 \in \Gamma T_1^*$ and and $\omega_2 \in \Gamma T_2^*$ and define

$$\eta^m := -\sigma_1 \wedge d\sigma_1 + \epsilon^2 \sigma_2 \wedge d\sigma_2 + \omega \sigma_1 \wedge \sigma_2.$$

The η^m is closed if and only if

$$0 = d\eta^m = -d\sigma_1 \curlywedge d\sigma_1 + \epsilon^2 d\sigma_2 \curlywedge d\sigma_2 + d\omega\, \sigma_1 \wedge \sigma_2 - \omega \wedge d(\sigma_1 \wedge \sigma_2). \tag{4.1}$$

Now let $\alpha, \beta \in \Gamma T_1^*$ and $\gamma, \delta \in \Gamma T_2^*$ such that

$$d_1\sigma_1 = \alpha\sigma_1 + \beta\sigma_2 \quad \text{and} \quad d_2\sigma_2 = \gamma\sigma_1 + \delta\sigma_2.$$

Remark 4.10. *Obviously, in the case that $\epsilon = 0$, q_1 and thus our lift σ_2 of s_2 may be chosen arbitrarily. To simplify the following analysis, we will fix q_1 by choosing a lift σ_2 so that $\delta = 0$. Note that this choice is unique up to multiplication by a smooth function g such that $d_2 g = 0$.*

Therefore, (4.1) is equivalent to

$$\begin{aligned} 0 &= -2\alpha \wedge (\sigma_1 \wedge d_2\sigma_1) + 2\epsilon^2 \delta \wedge (\sigma_2 \wedge d_1\sigma_2) + (2\epsilon^2\gamma - \omega_2) \wedge (\sigma_1 \wedge d_1\sigma_2) \\ &+ (-2\beta + \omega_1) \wedge (\sigma_2 \wedge d_2\sigma_1) + (d\omega - \omega_1 \wedge \delta - \omega_2 \wedge \alpha)\sigma_1 \wedge \sigma_2. \end{aligned} \tag{4.2}$$

Therefore, η^m is closed if and only if the following two conditions hold:

1. $\alpha = \delta = 0$, that is, $d_1\sigma_1 \in \Gamma T_1^* \otimes s_2$ and $d_2\sigma_2 \in \Gamma T_2^* \otimes s_1$.
2. $\omega = 2(\beta + \epsilon^2\gamma)$ and ω is closed.

We will now show that these two conditions can be reformulated as conditions on q.

Lemma 4.11. *q is divergence free with respect to the conformal structure c on $T\Sigma$ if and only if $d_1\sigma_1 \in \Gamma T_1^* \otimes s_2$ and $\epsilon^2 d_2\sigma_2 \in \Gamma T_2^* \otimes s_1$.*

Proof. Let g be a representative metric of the conformal structure c and let ∇ denote the Levi-Civita connection of g. Since T_1, T_2 are the maximally isotropic subbundles of this metric, we have that $\nabla_Z X \in \Gamma T_1$ and $\nabla_Z Y \in \Gamma T_2$ for any $X \in \Gamma T_1$, $Y \in \Gamma T_2$ and $Z \in \Gamma T\Sigma$.

Let $Q \in \Gamma End(T\Sigma)$ such that

$$q(X,Y) = g(X, Q(Y)).$$

Since $q \in \Gamma((T_1^*)^2 \oplus (T_2^*)^2)$, we have that $Q(T_1) \leq T_2$ and $Q(T_2) \leq T_1$. Hence, Q is symmetric and trace-free with respect to g. Now for $X \in \Gamma T_1$ and $Y \in \Gamma T_2$

$$\begin{aligned} d_Y(q(X,X)) &= -\epsilon^2 d_Y(d_X\sigma_2, d_X\sigma_2) \\ &= -2\epsilon^2(d_Y d_X\sigma_2, d_X\sigma_2) \\ &= -2\epsilon^2((d_X d_Y\sigma_2, d_X\sigma_2) + (d_{[Y,X]}\sigma_2, d_X\sigma_2)). \end{aligned}$$

On the other hand, since ∇ is the Levi-Civita connection we have that

$$d_Y(q(X,X)) = d_Y(g(X, Q(X))) = g(\nabla_Y X, Q(X)) + g(X, \nabla_Y Q(X)).$$

Furthermore,

$$-\epsilon^2(d_{[Y,X]}\sigma_2, d_X\sigma_2) = q([Y,X],X) = g(Q[Y,X],X) = g(Q(\nabla_Y X - \nabla_X Y), X) = g(Q(\nabla_Y X), X),$$

since $Q(\nabla_X Y) \in \Gamma T_1$. Therefore,

$$-2\epsilon^2(d_X d_Y\sigma_2, d_X\sigma_2) = g(\nabla_Y Q(X) - Q(\nabla_Y X), X).$$

Therefore, since $\nabla_Y Q(X) - Q(\nabla_Y X) \in \Gamma T_2$, $-2\epsilon^2(d_X d_Y\sigma_2, d_X\sigma_2) = 0$ if and only if $(\nabla_Y Q)(X) = 0$. One can then check that $-2\epsilon^2(d_X d_Y\sigma_2, d_X\sigma_2) = 0$ if and only if $\epsilon^2 d_Y\sigma_2 \in \Gamma T_2^* \otimes s_1$. Similarly, one can show that $(\nabla_X Q)(Y) = 0$ if and only if $d_X\sigma_1 \in \Gamma T_1^* \otimes s_2$. Therefore,

$$(d^\nabla Q)(X,Y) = (\nabla_X Q)(Y) - (\nabla_Y Q)(X) = 0$$

if and only if $\epsilon^2 d_2\sigma_2 \in \Gamma T_2^* \otimes s_1$ and $d_1\sigma_1 \in \Gamma T_1^* \otimes s_2$. The result follows by applying Corollary 2.29. □

Recall that we defined the Darboux cubic form $\mathcal{C} \in \Gamma S^3 T^*\Sigma \otimes (\wedge^2 f)^*$ as

$$\mathcal{C}(X,Y,Z)\sigma \wedge \nu = (\mathcal{D}_X\mathcal{D}_Y\sigma, \mathcal{N}_Z\nu) - (\mathcal{D}_X\mathcal{D}_Y\nu, \mathcal{N}_Z\sigma),$$

where $\sigma, \nu \in \Gamma f$ and $X, Y, Z \in \Gamma T\Sigma$. Thus, in terms of our special lifts σ_1 and σ_2 of the curvature spheres we have that

$$\begin{aligned} \mathcal{C}(X,Y,Z)\sigma_1 \wedge \sigma_2 &= -\gamma(Z)(d_Y\sigma_1, d_X\sigma_1) + \beta(Z)(d_Y\sigma_2, d_X\sigma_2) \\ &= -\gamma(Z) q_2(X,Y) + \beta(Z) q_1(X,Y) \end{aligned}$$

Therefore, from the Darboux cubic form $\mathcal{C}$, q determines a one form $\mathcal{C}^q$: let $X \in \Gamma T_1$, $Y \in \Gamma T_2$ such that

$$q_1(X,X) = q_2(Y,Y) = 1.$$

Let

$$\mathcal{C}^q := (\mathcal{C}(X,X,.) - \epsilon^2\mathcal{C}(Y,Y,.))\sigma_1 \wedge \sigma_2.$$

Then $\mathcal{C}^q = \beta + \epsilon^2\gamma$ and therefore condition 2 is equivalent to

$$d\mathcal{C}^q = 0.$$

Remark 4.12. *In the case that $\epsilon = 0$, X and thus $\mathcal{C}^q$ are determined by our choice of lift of s_2 in Remark 4.10. A different choice of such a lift scales $\mathcal{C}^q$ by a function g satisfying $d_2 g = 0$. Therefore, the closure of $\mathcal{C}^q$ is not affected by this choice.*

We have thus arrived at the following theorem:

Theorem 4.13. *f is an Ω-surface (Ω_0-surface) if and only if there exists a non-zero divergence free, non-degenerate (degenerate) quadratic differential (with respect to the conformal structure c induced by f) q satisfying*

$$d\mathcal{C}^q = 0.$$

Remark 4.14. *Condition 2 also tells us that the middle connection is given by*

$$\eta^m = \sigma_1 \wedge \star d\sigma_1 + \epsilon^2 \sigma_2 \wedge \star d\sigma_2,$$

where $\star$ is the hodge star operator induced by the conformal structure c.

4.2.1 Demoulin's equation

Suppose that $\mathfrak{q}, \mathfrak{p} \in \mathbb{R}^{4,2}$ define a space form vector and point sphere complex for a space form $\mathfrak{Q}^3$. Let $\mathfrak{f} \in \Gamma f$ be the projection of f into $\mathfrak{Q}^3$ and let $\mathfrak{t} \in \Gamma f$ be the corresponding tangent plane congruence. Then

$$\mathfrak{t} + \kappa_1 \mathfrak{f} \quad \text{and} \quad \mathfrak{t} + \kappa_2 \mathfrak{f}$$

are lifts of the curvature spheres s_1 and s_2, respectively, where κ_1 and κ_2 are the principal curvatures of $\mathfrak{f}$. Suppose that q is a non-zero divergence free quadratic differential form. Now there exists smooth functions λ and μ such that

$$\sigma_1 := \lambda(\mathfrak{t} + \kappa_1 \mathfrak{f}) \quad \text{and} \quad \sigma_2 := \mu(\mathfrak{t} + \kappa_2 \mathfrak{f})$$

are lifts of the curvature spheres such that

$$q = -\epsilon^2 (d\sigma_2, d\sigma_2) + (d\sigma_1, d\sigma_1),$$

where $\epsilon \in \{0, 1, i\}$. Since q is divergence free, we may choose curvature line coordinates (u, v) such that $q = -\epsilon^2 du^2 + dv^2$. Let $X := \frac{\partial}{\partial u}$ and $Y := \frac{\partial}{\partial v}$ and define

$$E := (d_X \mathfrak{f}, d_X \mathfrak{f}) \quad \text{and} \quad G := (d_Y \mathfrak{f}, d_Y \mathfrak{f}).$$

Then

$$\mu^2(\kappa_1-\kappa_2)^2E=(d_X\sigma_2,d_X\sigma_2)=1=(d_Y\sigma_1,d_Y\sigma_1)=\lambda^2(\kappa_1-\kappa_2)^2G. \tag{4.3}$$

We also have that

$$d_1\sigma_1=\beta\sigma_2 \quad \text{and} \quad d_2\sigma_2=\gamma\sigma_2.$$

Therefore,

$$\beta\mu(\mathfrak{t}+\kappa_2\mathfrak{f})=d_1\lambda\mathfrak{t}+((d_1\lambda)\kappa_1+\lambda(d_1\kappa_1))\mathfrak{f}.$$

Thus,

$$\beta=-\frac{\lambda}{\mu}\frac{d_1\kappa_1}{\kappa_1-\kappa_2}.$$

Similarly,

$$\gamma=\frac{\mu}{\lambda}\frac{d_2\kappa_2}{\kappa_1-\kappa_2}.$$

Now using (4.3) one can determine that

$$\mathcal{C}^q=\beta+\epsilon^2\gamma=\pm\left(-\frac{\sqrt{E}}{\sqrt{G}}\frac{d_1\kappa_1}{\kappa_1-\kappa_2}+\epsilon^2\frac{\sqrt{G}}{\sqrt{E}}\frac{d_2\kappa_2}{\kappa_1-\kappa_2}\right).$$

Therefore, in terms of our curvature line coordinates (u,v) we then have that

$$\mathcal{C}^q=\pm\left(-\frac{\sqrt{E}}{\sqrt{G}}\frac{\kappa_{1,u}}{\kappa_1-\kappa_2}du+\epsilon^2\frac{\sqrt{G}}{\sqrt{E}}\frac{\kappa_{2,v}}{\kappa_1-\kappa_2}dv\right).$$

Then $\mathcal{C}^q$ is closed if and only if

$$0=\left(\frac{\sqrt{E}}{\sqrt{G}}\frac{\kappa_{1,u}}{\kappa_1-\kappa_2}\right)_v+\epsilon^2\left(\frac{\sqrt{G}}{\sqrt{E}}\frac{\kappa_{2,v}}{\kappa_1-\kappa_2}\right)_u.$$

In terms of arbitrary curvature line coordinates (u,v), we then have that $q=-\epsilon^2U^2du^2+V^2dv^2$ and the closedness of $\mathcal{C}^q$ is equivalent to

$$0=\left(\frac{V}{U}\frac{\sqrt{E}}{\sqrt{G}}\frac{\kappa_{1,u}}{\kappa_1-\kappa_2}\right)_v+\epsilon^2\left(\frac{U}{V}\frac{\sqrt{G}}{\sqrt{E}}\frac{\kappa_{2,v}}{\kappa_1-\kappa_2}\right)_u, \tag{4.4}$$

where U is a function of u and V is a function of v and E and G are the coefficients of the first fundamental form of $\mathfrak{f}$ with respect to (u,v). Therefore, by Theorem 4.13, f is an Ω-/Ω_0-surface if and only if (4.4) is satisfied for some functions U and V, i.e., $\mathfrak{f}$ is an Ω-/Ω_0-surface in the sense of Demoulin [36].

4.2.2 Isothermic sphere congruences

In this subsection we will recall from [11, 50] the definition of isothermic sphere congruences and show that Ω-surfaces are the surfaces that envelop a pair of isothermic sphere congruences that separate the curvature spheres harmonically. Furthermore, we shall show that Ω_0-surfaces are the surfaces with a curvature sphere congruence that is isothermic.

Definition 4.15 ([11, 50])**.** *We say that a sphere congruence $s : \Sigma \to \mathbb{P}(\mathcal{L})$ is isothermic if there exists a non-zero closed one form $\eta \in \Omega^1(s \wedge s^\perp)$.*

By Remark 4.14, if f is Lie-applicable with quadratic differential q and σ_1 and σ_2 are special lifts of the curvature spheres such that

$$q = -\epsilon^2(d\sigma_2, d\sigma_2) + (d\sigma_1, d\sigma_1),$$

then the middle connection associated to q is given by

$$\eta^m = \sigma_1 \wedge \star d\sigma_1 + \epsilon^2 \sigma_2 \wedge \star d\sigma_2.$$

We may then gauge η^m by $\pm\epsilon\sigma_1 \wedge \sigma_2$ to obtain

$$\eta^\pm := \eta^m \pm d(\epsilon\sigma_1 \wedge \sigma_2) = (\sigma_1 \pm \epsilon\sigma_2) \wedge \star d(\sigma_1 \pm \epsilon\sigma_2) \in \Omega^1(s^\pm \wedge (s^\pm)^\perp), \tag{4.5}$$

where $s^\pm := \langle \sigma_1 \pm \epsilon\sigma_2 \rangle$. Thus, $s^\pm$ are isothermic sphere congruences. In the case that $\epsilon \neq 0$, $s^\pm$ are distinct and separate the curvature spheres harmonically. In the case that f is an Ω_0-surface, i.e., $\epsilon = 0$, we have that s_1 is an isothermic curvature sphere congruence and in fact the middle connection lies in $\Omega^1(s_1 \wedge s_1^\perp)$.

Theorem 4.16. *If f is an Ω-surface then f envelops two isothermic sphere congruences that separate the curvature spheres harmonically. Furthermore, if q is positive definite then these sphere congruences are complex conjugate and if q is indefinite then they are real.*

If f is an Ω_0-surface then f has a curvature sphere congruence that is isothermic.

Remark 4.17. *Notice that $\eta^m = \frac{1}{2}(\eta^+ + \eta^-)$. This is our justification for calling η^m the middle connection.*

Lemma 4.18. *Let $s \leq f$ and suppose that there exists $\eta \in [\eta^m]$ such that at a point $p \in \Sigma$*

$$\eta_p \in T_p^*\Sigma \otimes (s(p) \wedge f(p)^\perp).$$

Then s coincides with one of the isothermic sphere congruences enveloped by f at p.

Proof. Since $\eta \in [\eta^m]$, there exists a smooth function λ such that

$$\eta = \eta^m + d(\lambda\sigma_1 \wedge \sigma_2).$$

Now using that

$$\eta^m = \sigma_1 \wedge \star d\sigma_1 + \epsilon^2 \sigma_2 \wedge \star d\sigma_2$$

we have that

$$\eta = -\sigma_1 \wedge (\lambda d_1\sigma_2 - d_2\sigma_1) + \sigma_2 \wedge (\epsilon^2 d_1\sigma_2 - \lambda d_2\sigma_1)\, mod\, \Omega^1(\wedge^2 f).$$

Since $d_1\sigma_2$ and $d_2\sigma_1$ are linearly independent, η nowhere takes values in $s_2 \wedge f^{\perp}$, for all smooth functions λ. Therefore, let $\sigma(p) = \sigma_1(p) + \mu\sigma_2(p)$ be a lift of $s(p)$. Then

$$\eta_p \in T_p^*\Sigma \otimes (s(p) \wedge f(p)^{\perp})$$

if and only if

$$\mu(\lambda(p)d_1\sigma_2 - d_2\sigma_1) = \epsilon^2 d_1\sigma_2 - \lambda(p)d_2\sigma_1.$$

Since $d_1\sigma_2$ and $d_2\sigma_1$ are linearly independent at p, this is equivalent to

$$\mu = \lambda(p) \quad \text{and} \quad \lambda(p)^2 = \epsilon^2.$$

Thus, $\sigma(p) = \sigma_1(p) \pm \epsilon\sigma_2(p) \in s^{\pm}(p)$. □

The Δ_q operator

Suppose that $\epsilon \neq 0$ and let $X \in \Gamma T_1$ and $Y \in \Gamma T_2$ such that

$$q(X, X) = -\epsilon^2 \quad \text{and} \quad q_2(Y, Y) = 1.$$

Then we define an operator

$$\Delta_q := d_X d_X - \epsilon^2 d_Y d_Y.$$

Using Δ_q we define a map $\zeta_q : f \otimes f \to \mathbb{R}$ by

$$\zeta_q(\nu, \xi) = (\Delta_q \nu, \xi).$$

Lemma 4.19. *ζ_q is a symmetric tensor.*

Proof. The tensorial nature of ζ_q follows from the fact that $\Delta_q(\lambda\nu) = \lambda\Delta_q\nu\, mod\, f^{\perp}$ for $\nu \in \Gamma f$ and a smooth function λ. The symmetry of ζ_q follows from

$$(d_Z d_Z \nu, \xi) = -(d_Z \nu, d_Z \xi) = (\nu, d_Z d_Z \xi),$$

for any $Z \in \Gamma T\Sigma$, since $f^{(1)} = f^{\perp}$. □

Proposition 4.20. *Let $s \leq f$. Then $\zeta_q(s(p), s(p)) = 0$ if and only if s coincides with one of the isothermic sphere congruences at p.*

Proof. Let σ_1 and σ_2 be the special lifts of the curvature spheres s_1 and s_2, respectively, such that

$$q_1 = (d\sigma_2, d\sigma_2) \quad \text{and} \quad q_2 = (d\sigma_1, d\sigma_1).$$

Since s_1 and s_2 are curvature spheres, we have that

$$\Delta_q \sigma_1 = -\epsilon^2 d_Y d_Y \sigma_1 \, mod \, f^\perp \quad \text{and} \quad \Delta_q \sigma_2 = d_X d_X \sigma_2 \, mod \, f^\perp.$$

Let $\sigma \in \Gamma s$ and let α and β be smooth functions such that $\sigma = \alpha\sigma_1 + \beta\sigma_2$. Then

$$\zeta_q(\sigma, \sigma) = \beta^2 (d_X d_X \sigma_2, \sigma_2) - \epsilon^2 \alpha^2 (d_Y d_Y \sigma_1, \sigma_1) = \beta^2 - \epsilon^2 \alpha^2.$$

Thus, $\zeta_q(\sigma, \sigma) = 0$ if and only if $\beta = \pm\epsilon\alpha$, which holds if and only if $\sigma \in \Gamma s^\pm$. By Lemma 4.19, ζ_q is tensorial, thus this is a pointwise condition. □

Remark 4.21. *Δ_q and ζ_q will be useful later in the study of Darboux transformations.*

Christoffel dual lifts

Suppose that $\epsilon \neq 0$. Now $\mathcal{C}^q$ is closed, so there exists non-zero functions $\xi^\pm$ (unique up to constant scaling) such that

$$d\xi^\pm = \mp \epsilon^{-1} \mathcal{C}^q \xi^\pm.$$

We have that $\xi^+\xi^-$ is constant and, without loss of generality, we will assume that $\xi^+\xi^- = -1$. We then define lifts (unique up to reciprocal rescaling by a constant) $\sigma^\pm$ of the isothermic sphere congruences $s^\pm$, called Christoffel dual lifts of $s^\pm$, by

$$\sigma^\pm := \xi^\pm(\sigma_1 \pm \epsilon\sigma_2).$$

Proposition 4.22. *The isothermic connections are given by*

$$d + t\eta^\pm = \exp(t\tau^\pm) \cdot (d + t\eta^m) \tag{4.6}$$

where $\tau = \pm\frac{1}{2}\sigma^+ \wedge \sigma^-$. Furthermore, $\eta^\pm = \sigma^\pm \wedge d\sigma^\mp$ and $d\sigma^+ \curlywedge d\sigma^- = 0$.

Proof. By (4.5), we have that

$$\eta^\pm = \eta^m \pm d(\epsilon\sigma_1 \wedge \sigma_2) = \eta^m \mp d(\tfrac{1}{2}\sigma^+ \wedge \sigma^-).$$

Then (4.6) follows by Proposition 3.48. From (4.5) we can also deduce that

$$\eta^+ = (\sigma_1 + \epsilon\sigma_2) \wedge \star d(\sigma_1 + \epsilon\sigma_2) = -(\sigma_1 + \epsilon\sigma_2) \wedge d(\sigma_1 - \epsilon\sigma_2) - 2\mathcal{C}^q \sigma_1 \wedge \sigma_2.$$

On the other hand,

$$\begin{aligned}\sigma^+ \wedge d\sigma^- &= \xi^+\xi^-(\sigma_1 + \epsilon\sigma_2) \wedge d(\sigma_1 - \epsilon\sigma_2) + \xi^+ d\xi^-(\sigma_1 + \epsilon\sigma_2) \wedge (\sigma_1 - \epsilon\sigma_2) \\ &= -(\sigma_1 + \epsilon\sigma_2) \wedge d(\sigma_1 - \epsilon\sigma_2) + 2\xi^+\xi^-\mathcal{C}^q\sigma_1 \wedge \sigma_2 \\ &= -(\sigma_1 + \epsilon\sigma_2) \wedge d(\sigma_1 - \epsilon\sigma_2) - 2\mathcal{C}^q\sigma_1 \wedge \sigma_2.\end{aligned}$$

Hence, $\eta^+ = \sigma^+ \wedge d\sigma^-$. Similarly, one can check that $\eta^- = \sigma^- \wedge d\sigma^+$. The fact that

$$d\sigma^+ \curlywedge d\sigma^- = 0$$

follows from the closedness of η^+ (or η^-). $\square$

Remark 4.23. *We may also characterise Christoffel dual lifts in another way: let $V^\pm$ be the central sphere congruences of $s^\pm$, i.e.,*

$$V^\pm = (s^\pm)^{(1)} \oplus \langle \Delta(\sigma_1 \pm \epsilon\sigma_2)\rangle.$$

This yields splittings of $\underline{\mathbb{R}}^{4,2} = V^\pm \oplus (V^\pm)^\perp$ and thus induces splittings of the trivial connection. One can easily check that $s^\mp \leq (V^\pm)^\perp$ and furthermore the Christoffel lifts $\sigma^\mp$ are parallel sections of the aforementioned induced connections.

4.3 Associate surfaces

Let us recall the definition of O-surfaces given in [65]: suppose that $\nu^1, ..., \nu^n : \Sigma \to \mathbb{R}^3$ are Combescure transformations[1] of each other and let the subbundles $T_1, T_2 \leq T\Sigma$ denote the induced curvature subbundles on $T\Sigma$. Let κ_1^i and κ_2^i denote the principal curvatures of ν^i along T_1 and T_2, respectively, and define row vectors

$$K_j := (1/\kappa_j^1, ..., 1/\kappa_j^n),$$

for $j \in \{1, 2\}$. Then we say that $\{\nu^1, ..., \nu^n\}$ is a system of O-surfaces if there exists a constant symmetric $n \times n$ matrix S such that

$$K_1 S K_2^t = 0.$$

In this section we shall see how a system of O-surfaces arises from an Ω-surface.

In [34], Demoulin defines an associate surface of an Ω-surface: suppose that $\nu : \Sigma \to \mathbb{R}^3$ is an Ω-surface and in terms of curvature line coordinates (u, v) the third fundamental form of ν is given by $III = p^2 du^2 + r^2 dv^2$. Then there exists a Combescure transformation $\nu^D : \Sigma \to \mathbb{R}^3$

[1] That is, the curvature directions of ν^i are parallel to the curvature directions of ν^j for all $i, j \in \{1, ..., n\}$.

of ν and there exist functions U of u and V of v such that

$$\left(\frac{1}{\kappa_1}-\frac{1}{\kappa_2}\right)\left(\frac{1}{\kappa_1^D}-\frac{1}{\kappa_2^D}\right)=-\epsilon^2\frac{U^2}{p^2}+\frac{V^2}{r^2}, \tag{4.7}$$

where κ_1 and κ_2 denote the principal curvatures of ν, κ_1^D and κ_2^D denote the principal curvatures of ν^D and $\epsilon\in\{1,i\}$. Conversely, if two surfaces are in such a relation then they are Ω-surfaces.

Suppose that $f:\Sigma\to\mathcal{Z}$ is an Ω-surface. Then there exists a closed one-form $\eta\in\Omega^1(f\wedge f^\perp)$ such that the quadratic differential associated to η is non-degenerate. Let $\mathfrak{q}_\infty$ and $\mathfrak{p}$ be a space form vector and point sphere complex with $|\mathfrak{q}_\infty|^2=0$, i.e.,

$$\mathfrak{Q}^3:=\{y\in\mathcal{L}:(y,\mathfrak{q}_\infty)=-1,(y,\mathfrak{p})=0\}$$

has sectional curvature $\kappa=0$. Then we may choose a null vector $\mathfrak{q}_0\in\langle\mathfrak{p}\rangle^\perp$ such that $(\mathfrak{q}_0,\mathfrak{q}_\infty)=-1$. Thus $\langle\mathfrak{q}_\infty,\mathfrak{p},\mathfrak{q}_0\rangle^\perp\cong\mathbb{R}^3$ and we have an isometry

$$\phi:\langle\mathfrak{q}_\infty,\mathfrak{p},\mathfrak{q}_0\rangle^\perp\to\mathfrak{Q}^3,\quad x\mapsto x+\mathfrak{q}_0+\frac{1}{2}(x,x)\mathfrak{q}_\infty.$$

We can use this to identify $\mathfrak{f}:=f\cap\mathfrak{Q}^3$ with $\nu:\Sigma\to\mathbb{R}^3$. We then have that $d\mathfrak{f}=d\nu+(d\nu,\nu)\mathfrak{q}_\infty$ and $\mathfrak{t}=n+(n,\nu)\mathfrak{q}_\infty+\mathfrak{p}$.

Now $(\eta\mathfrak{p},\mathfrak{q}_\infty)$ is a closed one-form, so there exists (up to addition of a constant) $\lambda:\Sigma\to\mathbb{R}$ such that $d\lambda=(\eta\mathfrak{p},\mathfrak{q}_\infty)$. Then we may gauge η by $\tau:=-\lambda\mathfrak{f}\wedge\mathfrak{t}$ to obtain $\tilde{\eta}$ with $(\tilde{\eta}\mathfrak{p},\mathfrak{q}_\infty)=0$. Therefore, we shall assume that $(\eta\mathfrak{p},\mathfrak{q}_\infty)=0$. From this we can deduce that η is of the form

$$\eta=\mathfrak{f}\wedge d\mathfrak{f}\circ A+\mathfrak{t}\wedge d\mathfrak{t}\circ B,$$

for some $A,B\in End(T\Sigma)$. The closure of η implies that $\eta\mathfrak{q}_\infty=-d\mathfrak{f}\circ A$ and $\eta\mathfrak{p}=-d\mathfrak{t}\circ B$ are closed and that

$$d\mathfrak{f}\curlywedge d\mathfrak{f}\circ A+d\mathfrak{t}\curlywedge d\mathfrak{t}\circ B=0. \tag{4.8}$$

The closure of $d\mathfrak{f}\circ A$ implies that $d\nu\circ A$ is closed. Furthermore, by Lemma 3.34, we have that $\eta(T_i)\leq f\wedge f_i$ and thus

$$A\in\Gamma(T_1^*\otimes T_1\oplus T_2^*\otimes T_2).$$

Therefore, locally there exists $\nu^D:\Sigma\to\mathbb{R}^3$ such that $d\nu^D=d\nu\circ A$ and ν^D has parallel parallel curvature directions to ν. Similarly there exists $\hat{\nu}:\Sigma\to\mathbb{R}^3$ such that $d\hat{\nu}=dn\circ B$ and $B\in\Gamma(T_1^*\otimes T_1\oplus T_2^*\otimes T_2)$. Thus, $\hat{\nu}$ also has parallel parallel curvature directions to ν. From equation (4.8) and Rodrigues' equation, we can then deduce that

$$\frac{1}{\kappa_1\kappa_2^D}+\frac{1}{\kappa_2\kappa_1^D}-\frac{1}{\hat{\kappa}_1}-\frac{1}{\hat{\kappa}_2}=0. \tag{4.9}$$

Conversely, given Combescure transformations ν^D and $\hat{\nu}$ of ν such that (4.9) is satisfied we may define a closed one form

$$\eta = \mathfrak{f} \wedge (d\nu^D + (d\nu^D, \nu)\mathfrak{q}_\infty) + \mathfrak{t} \wedge (d\hat{\nu} + (d\hat{\nu}, \nu)\mathfrak{q}_\infty).$$

Theorem 4.24. *ν is an Ω-surface if and only if there exists an associate surface ν^D and an associate Gauss map $\hat{\nu}$ that are Combescure transformations of ν such that the principal curvatures of ν, ν^D and $\hat{\nu}$ satisfy (4.9).*

Remark 4.25. *We shall assume that ν^D and $\hat{\nu}$ are oriented so that the Gauss map of these surfaces is $-n$.*

Remark 4.26. *In Theorem 4.24, we allow ν^D and $\hat{\nu}$ to degenerate to be points. In which case we consider their principal radii of curvature to be zero. We shall see in Subsection 6.3.1 that $\hat{\nu}$ degenerating to a point is a characterisation of isothermic surfaces, in which case ν^D becomes the Christoffel transform of ν. We will also see in Subsection 6.3.3 that ν^D degenerating to a point is a characterisation of L-isothermic surfaces in which case $\hat{\nu}$ becomes a minimal surface.*

Remark 4.27. *The addition of a constant c to λ sends*

$$\nu^D \mapsto \nu^D + cn \quad \textit{and} \quad \hat{\nu} \mapsto \hat{\nu} - c\nu.$$

Thus we get a parallel surface to ν^D. The behaviour of $\hat{\nu}$ under this change is our motivation for calling $\hat{\nu}$ an associate Gauss map.

Remark 4.28. *If we let*

$$S = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

then one can see that condition (4.9) shows that $\{\nu, \nu^D, \hat{\nu}, n\}$ is a system of O-surfaces, where we consider the Gauss map n to be oriented so that its principal curvatures are both -1.

We also have that the quadratic form of η is given by

$$q = -(d\nu, d\nu^D) - (dn, d\hat{\nu}).$$

On the other hand, in terms of curvature line coordinates (u, v), we have that

$$q = -\epsilon^2 U^2 du^2 + V^2 dv^2,$$

for some functions U of u and V of v. Hence,

$$-\epsilon^2 U^2 = \left(\frac{1}{\kappa_1 \kappa_1^D} - \frac{1}{\hat{\kappa}_1} \right) p^2 \quad \text{and} \quad V^2 = \left(\frac{1}{\kappa_2 \kappa_2^D} - \frac{1}{\hat{\kappa}_2} \right) r^2$$

and thus
$$\frac{1}{\hat{\kappa}_1} = \frac{\epsilon^2 U^2}{p^2} + \frac{1}{\kappa_1 \kappa_1^D} \quad \text{and} \quad \frac{1}{\hat{\kappa}_2} = -\frac{V^2}{r^2} + \frac{1}{\kappa_2 \kappa_2^D}.$$
Then substituting this into (4.9) yields (4.7). Hence, ν^D is an associate surface of ν, in the sense of [34].

We may write η as
$$\eta = Ad_{\exp(\nu \wedge \mathfrak{q}_\infty)}(\mathfrak{q}_0 \wedge d\nu^D + l \wedge d\hat{\nu}),$$
where $l := n + \mathfrak{p}$. By the symmetry of equation (4.9), ν^D is an Ω-surface with closed one-form
$$\eta^D := Ad_{\exp(\nu^D \wedge \mathfrak{q}_\infty)}(\mathfrak{q}_0 \wedge d\nu - l^D \wedge d\hat{\nu}),$$
where $l^D := -n + \mathfrak{p}$. Furthermore, the quadratic differential q^D defined by η^D agrees with q.

Theorem 4.29. *An associate surface of an Ω-surface is itself an Ω-surface.*

Remark 4.30. *In [34], Demoulin used the Christoffel transform of isothermic surfaces in $\mathbb{R}^{3,1}$ to derive the associate surface. This leads the author to believe that the associate surface is a Laguerre geometric object, however it is unclear what the associate Gauss map is in this setting.*

Chapter 5

Transformations of Ω-/Ω_0-surfaces

One of the main motivations for studying Ω- and Ω_0-surfaces is their rich transformation theory. In [38, 39], Eisenhart developed a transformation for Ω-surfaces, analogous to the Darboux transformation for isothermic surfaces. In [28], Clarke gave a modern treatment of the transformation theory of Lie applicable surfaces. In a similar way to isothermic surfaces, [8, 11, 18, 50, 64], the richness of this theory follows from the flatness of a one-parameter family of connections. In this chapter we shall review and expand on this theory.

Remark 5.1. *It should be remarked that the transformation theory presented here is intimately linked with loop group methods. In [70], Terng and Uhlenbeck used loop groups to explain the existence of Bäcklund transforms of soliton equations. This theory is generalised in [69] and in [28, Section 1.1.3] Clarke discusses how this relates to the transformation theory of Lie applicable surfaces. However, we shall not expand on this link any further here.*

Let $\mathcal{Z}$ denote the space of lines in $\mathbb{R}^{s,t}$ and suppose that $f : \Sigma \to \mathcal{Z}$ is an applicable Legendre map. Then by Corollary 3.45, there exists a non-trivial closed one form $\eta \in \Omega^1(f \wedge f^\perp)$ whose quadratic differential q is non-zero. Recall that by Proposition 3.47 we have that $\{d^t := d + t\eta\}_{t\in\mathbb{R}}$ is a one-parameter family of flat metric connections and, by Proposition 3.48, if $\tilde{\eta}$ is in the gauge-orbit of η, i.e., there exists $\tau \in \Gamma(\wedge^2 f)$ such that $\tilde{\eta} = \eta - d\tau$, then

$$d + t\tilde{\eta} = \exp(t\tau) \cdot (d + t\eta). \tag{5.1}$$

5.1 Calapso transforms

Remark 5.2. *It should be noted that this transformation is commonly referred to as the spectral deformation.*

Since d^t is a flat metric connection for each $t \in \mathbb{R}$, there exist local trivialising orthogonal gauge transformations of d^t, that is $T(t) : \Sigma \to \mathrm{O}(s,t)$ such that

$$T(t) \cdot d^t = d.$$

Definition 5.3. *$(f^t)_{t\in\mathbb{R}}$, where $f^t := T(t)f$, are called the Calapso transforms of f.*

Lemma 5.4. *The Calapso transforms are invariant under gauge transformation.*

Proof. It follows from (5.1) that the local trivialising orthogonal gauge transformations for $d + t\tilde{\eta}$ are $\widetilde{T}(t) = T(t)\exp(-t\tau)$. Thus, $\widetilde{T}(t)f = T(t)f$. $\square$

Lemma 5.5. *The Calapso transforms are Legendre maps and $s^t := T(t)s$ is a curvature sphere of f^t if and only if s is a curvature sphere of f. Furthermore, the corresponding curvature subbundles coincide.*

Proof. Let $\sigma^t := T(t)\sigma$ be a section of f^t. Then

$$d\sigma^t = d(T(t)\sigma) = T(t)d^t\sigma = T(t)d\sigma, \tag{5.2}$$

since $\eta\sigma = 0$. Then as $d\sigma \in \Omega^1(f^\perp)$ and $T(t)$ takes values in $\mathrm{O}(s,t)$ we have that $d\sigma^t \in \Omega^1((f^t)^\perp)$. Hence, the contact condition holds for f^t. Furthermore, for $X_p \in T_p\Sigma$, it follows from (5.2) that $d_{X_p}\sigma^t \in f^t(p)$ if and only if $d_{X_p}\sigma \in f(p)$. Therefore, since the immersion condition holds for f, it holds for f^t as well. Moreover, we can deduce that X_p is a curvature direction of a curvature sphere $s^t := T(t)s$ if and only if X_p is a curvature direction of s. $\square$

Remark 5.6. *From Lemma 5.5 we can deduce that f^t has an umbilic at $p \in \Sigma$ if and only if f has an umbilic at p.*

Theorem 5.7. *$\eta^t := Ad_{T(t)} \cdot \eta$ is a closed one form with values in $\Omega^1(f^t \wedge (f^t)^\perp)$ with $q^t = q$. Hence, f^t is a Lie applicable surface.*

Proof. The closedness of η^t follows from

$$d\eta^t = (T(t) \cdot d^t)Ad_{T(t)} \cdot \eta = T(t) \cdot d^t\eta = T(t) \cdot (d\eta + t[\eta \wedge \eta]) = 0.$$

Furthermore, for $\sigma^t := T(t)\sigma$

$$\eta^t(X)d_Y\sigma^t = (Ad_{T(t)}\eta(X))(T(t) \cdot (d + t\eta)(Y))\sigma^t = T(t)\eta(X)d_Y\sigma.$$

Thus,

$$q^t(X,Y) = tr(\sigma^t \mapsto \eta^t(X)d_Y\sigma^t)$$

coincides with $q(X,Y)$ for all $X, Y \in \Gamma T\Sigma$. $\square$

We will now see how the one parameter family of flat connections of a Calapso transform are related to those of the original surface:

Proposition 5.8. *For any $s \in \mathbb{R}$,*

$$d + s\eta^t = T(t) \cdot (d + (s+t)\eta).$$

Therefore the local trivialising orthogonal gauge transformations of $d + s\eta^t$ are

$$T^t(s) = T(s+t)T^{-1}(t).$$

Proof. Using Theorem 5.7, we have that

$$d + s\eta^t = d + sAd_{T(t)} \cdot \eta = T(t) \cdot (T^{-1}(t) \cdot d + s\eta) = T(t) \cdot (d + (s+t)\eta).$$

$\square$

From Proposition 5.8 we can quickly deduce a result of [15], which is the analogue of the permutability result of Hertrich-Jeromin [50, Subsection 5.5.9] for Calapso transforms of isothermic surfaces:

Corollary 5.9.

$$T^t(s)T(t) = T(s+t).$$

Now we restrict our attention to $\mathbb{R}^{4,2}$ and consider the case where f has exactly two curvature spheres s_1 and s_2. Then by Lemma 5.5, f^t has exactly two curvature sphere $s_1^t = T(t)s_1$ and $s_2^t = T(t)s_2$. Furthermore the curvature subbundles T_1 and T_2 of these curvature spheres are preserved. Suppose now that η^m is the middle connection and $(T(t))_{t\in\mathbb{R}}$ are the corresponding gauge transformations.

Lemma 5.10. *The Lie cyclides of f^t are given by*

$$S_1^t = T(t)S_1 \quad \textit{and} \quad S_2^t = T(t)S_2.$$

Hence, the induced splitting of the trivial connection $d = \mathcal{D}^t + \mathcal{N}^t$ satisfies

$$\mathcal{D}^t = T(t) \cdot \mathcal{D} \quad \textit{and} \quad \mathcal{N}^t = T(t) \cdot \mathcal{N}.$$

Proof. Let $Y \in \Gamma T_2$ and $\sigma_1^t = T(t)\sigma_1$ be a lift of the curvature sphere s_1^t. Then

$$d_Y\sigma_1^t = d_Y(T(t)\sigma_1) = (T(t) \cdot d_Y^t)T(t)\sigma_1 = T(t)(d_Y^t\sigma_1) = T(t)d_Y\sigma_1,$$

since $\eta^m f = 0$. Thus, $d\sigma_1^t(T_2) = T(t)d\sigma_1(T_2)$. Furthermore,

$$d_Y d_Y\sigma_1^t = d_Y d_Y(T(t)\sigma_1) = (T(t) \cdot d_Y^t)(T(t)d_Y\sigma_1) = T(t)d_Y^t d_Y\sigma_1.$$

Now, since we are using the middle connection, $\eta^m(Y)d_Y\sigma_1 \in \Gamma s_1$. Thus, $d_Y d_Y\sigma_1^t \in \Gamma T(t)S_1$ and

$$S_1^t = s_1^t \oplus d\sigma_1^t(T_2) \oplus \langle d_Y d_Y\sigma_1^t\rangle = T(t)S_1.$$

Similarly, $S_2^t = T(t)S_2$. $\square$

Since the curvature subbundles T_1 and T_2 are preserved, we have that the conformal structure of Subsection 2.5.1 is preserved. In fact, since $\mathcal{N}^t = T(t) \cdot \mathcal{N}$, it is clear from (2.2) that a stronger statement is true:

Corollary 5.11. *The Lie-invariant metric is preserved by Calapso transform.*

Furthermore, one can deduce from Lemma 5.10 how the Darboux cubic form of Subsection 2.5.2 behaves:

Corollary 5.12. *The Darboux cubic form $\mathcal{C}^t \in \Gamma S^3 T\Sigma \otimes (\wedge^2 f^t)^*$ of f^t satisfies*

$$\mathcal{C}^t \circ T(t) = \mathcal{C}, \tag{5.3}$$

that is, for $\tau \in \Gamma(\wedge^2 f)$ and $X, Y, Z \in \Gamma T\Sigma$,

$$\mathcal{C}^t(X,Y,Z)(T(t)\cdot\tau) = \mathcal{C}(X,Y,Z)\tau.$$

Proof. By Lemma 5.10,

$$\mathcal{D}^t = T(t)\cdot\mathcal{D} \quad \text{and} \quad \mathcal{N}^t = T(t)\cdot\mathcal{N}.$$

The result then follows by observing that for $\nu, \xi \in \Gamma f$ and $X, Y, Z \in \Gamma T\Sigma$

$$\begin{aligned} (\mathcal{D}^t_X\mathcal{D}^t_Y(T(t)\nu), \mathcal{N}^t_Z(T(t)\xi)) &= ((T(t)\cdot\mathcal{D}_X)(T(t)\cdot\mathcal{D}_Y)T(t)\nu, (T(t)\cdot\mathcal{N}_Z)T(t)\xi) \\ &= (T(t)\mathcal{D}_X\mathcal{D}_Y\nu, T(t)\mathcal{N}_Z\xi) \\ &= (\mathcal{D}_X\mathcal{D}_Y\nu, \mathcal{N}_Z\xi), \end{aligned}$$

by the orthogonality of $T(t)$. $\square$

Remark 5.13. *Since $T(t)f$ is independent of gauge η, (5.3) holds for the gauge transformations of any choice of gauge potential η in the orbit of η^m.*

Remark 5.14. *Corollaries 5.11 and 5.12 should come as no surprise as Blaschke showed in [3] that the only surfaces that are not determined by the Lie-invariant metric and Darboux cubic form are the Lie-applicable surfaces.*

Corollary 5.15. *The middle connection of f^t is $(\eta^t)^m = Ad_{T(t)}\cdot\eta^m$.*

Proof. In Section 4.1 we had a splitting $\underline{\mathfrak{o}(4,2)} = \mathfrak{h} + \mathfrak{m}$ induced by f, where

$$\mathfrak{h} = S_1\wedge S_1 \oplus S_2 \wedge S_2 \quad \text{and} \quad \mathfrak{m} = S_1 \wedge S_2.$$

By Lemma 5.10, f^t induces the splitting $\underline{\mathfrak{o}(s,t)} = \mathfrak{h}^t + \mathfrak{m}^t$, where

$$\mathfrak{h}^t = T(t)\cdot\mathfrak{h} \quad \text{and} \quad \mathfrak{m}^t = T(t)\cdot\mathfrak{m}.$$

We may then split $\eta^m = \eta_{\mathfrak{h}} + \eta_{\mathfrak{m}}$, where $\eta_{\mathfrak{h}} \in \Omega^1(\mathfrak{h})$ and $\eta_{\mathfrak{m}} \in \Omega^1(\mathfrak{m})$. Now splitting $(\eta^t)^m = Ad_{T(t)} \cdot \eta^m$ with respect to the splitting induced by f^t yields $(\eta^t)^m = \eta^t_{\mathfrak{h}^t} + \eta^t_{\mathfrak{m}^t}$ with

$$\eta^t_{\mathfrak{h}^t} = Ad_{T(t)} \cdot \eta_{\mathfrak{h}} \quad \text{and} \quad \eta^t_{\mathfrak{m}^t} = Ad_{T(t)} \cdot \eta_{\mathfrak{m}}.$$

Since η^m is the middle connection, $\eta_{\mathfrak{m}} \in \Omega^1(f \wedge f)$. Hence,

$$\eta^t_{\mathfrak{m}^t} = Ad_{T(t)} \cdot \eta_{\mathfrak{m}} \in \Omega^1(f^t \wedge f^t).$$

Therefore $(\eta^t)^m$ is the middle connection of f^t. □

Proposition 5.16. *Suppose that s is an isothermic sphere congruence of f with isothermic gauge potential $\eta^s \in \Omega^1(s \wedge s^\perp)$. Then $s^t := T(t)s$ is an isothermic sphere congruence of f^t with isothermic gauge potential $(\eta^t)^s := Ad_{T(t)} \cdot \eta^s$.*

Proof. From the orthogonality of $T(t)$ we have

$$T(t) \cdot s \wedge s^\perp = s^t \wedge (s^t)^\perp.$$

Hence, $(\eta^t)^s \in \Omega^1(s^t \wedge (s^t)^\perp)$ and s^t is isothermic. □

Remark 5.17. *s^t is in fact the Calapso transform of the isothermic sphere congruence s, see [11, 50].*

5.2 Darboux transforms

Fix $m \in \mathbb{R}^\times$. Since d^m is a flat connection, it has many parallel sections. Suppose that $\hat{s}$ is a null rank one parallel subbundle of d^m such that $\hat{s}$ is nowhere orthogonal to the curvature sphere congruences of f. Let $s_0 := \hat{s}^\perp \cap f$ and let $\hat{f} := s_0 \oplus \hat{s}$.

Definition 5.18. *$\hat{f}$ is a Darboux transform of f with parameter m.*

We will now show that $\hat{f}$ is applicable.

Lemma 5.19. *For any section $\sigma_0 \in \Gamma s_0$ and any parallel section $\hat{\sigma} \in \Gamma\hat{s}$ of d^m*

$$d\sigma_0,\, d\hat{\sigma} \in \Omega^1((f + \hat{f})^\perp).$$

Proof. Let $\sigma_0 \in \Gamma s_0$. Then as f is a Legendre map we have that $d\sigma_0 \in \Omega^1(f^\perp)$. Now if $\hat{\sigma} \in \Gamma\hat{s}$ is a parallel section of d^m then $d\hat{\sigma} = -m\eta\hat{\sigma} \in \Omega^1(f^\perp)$. Furthermore, since $\hat{s}$ is null, $d\hat{\sigma} \in \Omega^1(\hat{s}^\perp)$. Thus,

$$d\hat{\sigma} \in \Omega^1((f + \hat{f})^\perp).$$

Finally, since $(\sigma_0, \hat{\sigma}) = 0$, by the Leibniz rule we have

$$(d\sigma_0, \hat{\sigma}) = -(\sigma_0, d\hat{\sigma}) = 0.$$

Thus,

$$d\sigma_0 \in \Omega^1((f+\hat{f})^\perp).$$

$\square$

Lemma 5.20. *$\hat{f}$ is a Legendre map.*

Proof. By Lemma 5.19 $\hat{f}$ satisfies the contact condition. It remains to check the immersion condition of $\hat{f}$: let $p \in \Sigma$ and suppose that there exists $X \in T_p\Sigma$ such that $d_X\sigma_0 \in \hat{f}(p)$ for some lift $\sigma_0 \in \Gamma s_0$. Then as $d\sigma_0 \in \Omega^1((f+\hat{f})^\perp)$, we have that $d_X\sigma_0 \in s_0(p)$. Then it follows from the fact that s_0 is nowhere a curvature sphere of f that $X = 0$. $\square$

Recall from Section 2.7 that we defined Ribaucour transforms of Legendre maps.

Lemma 5.21. *$\hat{f}$ is a Ribaucour transform of f.*

Proof. By Lemma 5.19, for a parallel section $\hat{\sigma} \in \Gamma\hat{s}$ of d^m,

$$d\hat{\sigma} \in \Omega^1((f+\hat{f})^\perp).$$

Therefore, $\hat{\sigma}\, mod\, s_0$ is a parallel section of the induced connection on $(f+\hat{f})/s_0$. Hence, this connection is flat. $\square$

Suppose that $s \le f$ is a rank one subbundle of f such that $s \cap s_0 = \{0\}$ and define $l := s \oplus \hat{s}$. Then l defines a $(1,1)$-subbundle of $\underline{\mathbb{R}}^{s,t}$ and we have the following splitting of $\underline{\mathbb{R}}^{s,t}$:

$$\underline{\mathbb{R}}^{s,t} = l \oplus l^\perp.$$

We can then use this splitting to split the trivial connection d on $\underline{\mathbb{R}}^{s,t}$ into

$$d = \mathcal{D}^{l,l^\perp} + \mathcal{N}^{l,l^\perp},$$

where $\mathcal{D}^{l,l^\perp}$ is the sum of the induced connections $\mathcal{D}^l$ and $\mathcal{D}^{l^\perp}$ on l and $l^\perp$, respectively, and $\mathcal{N}^{l,l^\perp} \in \Omega^1(l \wedge l^\perp)$. By Corollary 2.38, $\mathcal{D}^l$ is a flat connection on l and if $\hat{\sigma}$ is a parallel section of d^m, then $\hat{\sigma}$ is a parallel section of $\mathcal{D}^l$. We may further split $\mathcal{N}^{l,l^\perp} = -\beta - \hat{\beta}$ where

$$\beta \in \Omega^1(\hat{s} \wedge l^\perp) \quad \text{and} \quad \hat{\beta} \in \Omega^1(s \wedge l^\perp).$$

Moreover we may use our splitting to split $\eta = \eta_0 + \eta_s$, where

$$\eta_0 \in \Omega^1(s_0 \wedge l^\perp) \quad \text{and} \quad \eta_s \in \Omega^1(s \wedge l^\perp).$$

Recall from [18, 64, 11, 28] that for $v, w \in \mathcal{L}$ such that $(v,w) \neq 0$ and $t \in \mathbb{R}^\times$ we have an orthogonal transformation

$$\Gamma_w^v(t)u = \begin{cases} t\,v & \text{for } u = v, \\ \frac{1}{t}\,w & \text{for } u = w, \\ u & \text{for } u \in \langle v, w\rangle^\perp. \end{cases}$$

We are now in a position to state the following proposition:

Proposition 5.22. *There exists a closed one-form $\hat{\eta} \in \Omega^1(\hat{f} \wedge \hat{f}^\perp)$ such that*

$$d + t\hat{\eta} = \Gamma_s^{\hat{s}}(1 - t/m) \cdot (d + t\eta).$$

Furthermore, s is a parallel subbundle of $d + m\hat{\eta}$ and the quadratic differential $\hat{q}$ of $\hat{\eta}$ coincides with q.

Proof. The first part of this theorem was proved by Clarke [28, Theorem 4.3.7] and is analogous to [11, Proposition 3.11]. For the purpose of proving the latter part of this theorem, we shall repeat the arguments of those proofs here.

Firstly, for a parallel section $\hat{\sigma} \in \Gamma\hat{s}$ of d^m, we have that $d\hat{\sigma} = -m\eta\hat{\sigma}$. Therefore $-\hat{\beta}\hat{\sigma} = -m\eta_s\hat{\sigma}$. This implies that $\hat{\beta} = m\eta_s$. Now we may write

$$d + t\eta = \mathcal{D}^{l,l^\perp} - \beta - \hat{\beta} + t\eta_0 + t\eta_s.$$

Therefore,

$$\begin{aligned} \Gamma_s^{\hat{s}}(1 - t/m) \cdot (d + t\eta) &= \Gamma_s^{\hat{s}}(1 - t/m) \cdot \mathcal{D}^{l,l^\perp} + Ad_{\Gamma_s^{\hat{s}}(1-t/m)} \cdot (-\beta - (1 - t/m)\hat{\beta} + t\eta_0) \\ &= \mathcal{D}^{l,l^\perp} - (1 - t/m)\beta - (1 - t/m)/(1 - t/m)\hat{\beta} + t\eta_0 \\ &= \mathcal{D}^{l,l^\perp} - \hat{\beta} - \beta + t(\eta_0 + (1/m)\beta). \end{aligned}$$

Then letting $\eta_{\hat{s}} := (1/m)\beta$ and $\hat{\eta} := \eta_0 + \eta_{\hat{s}} \in \Omega^1(\hat{f} \wedge \hat{f}^\perp)$, we have that

$$d + t\hat{\eta} = \Gamma_s^{\hat{s}}(1 - t/m) \cdot (d + t\eta).$$

Since $d + t\eta$ is a one-parameter family of flat connections, we must have that $d + t\hat{\eta}$ is a one-parameter family of flat connections. The curvature of this family is given by

$$R^{d+t\hat{\eta}} = td\hat{\eta} + \frac{t^2}{2}[\hat{\eta} \wedge \hat{\eta}].$$

Thus, $\hat{\eta}$ is closed and $[\hat{\eta} \wedge \hat{\eta}] = 0$.

Suppose that $\sigma \in \Gamma s$ is a parallel section of $\mathcal{D}^l$. Then $d\sigma = -\beta\sigma$ and

$$(d + m\hat{\eta})\sigma = -\beta\sigma + m(1/m)\beta\sigma = 0.$$

Hence, s is a parallel subbundle of $d + m\hat{\eta}$.

We shall now show that the quadratic forms of $\hat{\eta}$ and η coincide: let $\sigma_0 \in \Gamma s_0$ and assume that $(\sigma, \hat{\sigma}) = -1$. Now, $\{\sigma_0, \sigma\}$ is a basis for f and $\{\sigma_0, \hat{\sigma}\}$ is a basis for $\hat{f}$. Since $\hat{\eta} = \eta_0 + \eta_{\hat{s}}$ and $\eta = \eta_0 + \eta_s$, we have for $X, Y \in \Gamma T\Sigma$,

$$[\hat{\eta}(X)d_Y\sigma_0]_{s_0} = [\eta_0(X)d_Y\sigma_0]_{s_0} = [\eta(X)d_Y\sigma_0]_{s_0}.$$

Therefore, with respect to our bases defined above, the s_0 component of $\hat{\eta}(X)d_Y\sigma_0$ coincides with the s_0 component of $\eta(X)d_Y\sigma_0$. Furthermore, the $\hat{s}$ component of $\hat{\eta}(X)d_Y\hat{\sigma}$ is given by

$$-(\hat{\eta}(X)d_Y\hat{\sigma}, \sigma) = (1/m)(\beta(X)\hat{\beta}(Y)\hat{\sigma}, \sigma) = (1/m)(\hat{\sigma}, \hat{\beta}(Y)\beta(X)\sigma) = -(\hat{\sigma}, \eta(Y)d_X\sigma),$$

by the skew-symmetry of β and $\hat{\beta}$. Therefore, the $\hat{s}$ component of $\hat{\eta}(X)d_Y\hat{\sigma}$ coincides with the s component of $\eta(X)d_Y\sigma$. It follows then that

$$q(X, Y) = tr(\nu \mapsto \eta(X)d_Y\nu) \quad \text{and} \quad \hat{q}(X, Y) = tr(\hat{\nu} \mapsto \hat{\eta}(X)d_Y\hat{\nu})$$

are equivalent. $\square$

As a corollary to Proposition 5.22 we have the following theorem:

Theorem 5.23. *$\hat{f}$ is an applicable hypersurface and f is a Darboux transform of $\hat{f}$ with parameter m.*

An obvious question to ask is what happens if we use a different gauge $\tilde{\eta} = \eta - d\tau$ to compute our Darboux transforms.

Lemma 5.24. *$\exp(m\tau)\hat{s} \leq \hat{f}$ is a parallel subbundle of $d + m\tilde{\eta}$.*

Proof. One can easily check that $\exp(m\tau)\hat{s} \leq \hat{f}$ and by (5.1) this is a parallel subbundle of $d + m\tilde{\eta}$. $\square$

Remark 5.25. *Lemma 5.24 shows that Darboux transforms are invariant of gauge transformation.*

Now suppose that $\tilde{l}$ is a general rank 2 subbundle of $f + \hat{f}$ such that $s_0 \cap \tilde{l} = \{0\}$. Let $s_{\tilde{l}} := f \cap \tilde{l}$ and $\hat{s}_{\tilde{l}} := \hat{f} \cap \tilde{l}$. Then Clarke showed in [28] that there exists $\tau_{\tilde{l}} \in \Gamma(\wedge^2 f)$ and $\hat{\tau}_{\tilde{l}} \in \Gamma(\wedge^2 \hat{f})$ such that

$$\hat{s}_{\tilde{l}} = \exp(m\tau_{\tilde{l}})\hat{s} \quad \text{and} \quad s_{\tilde{l}} = \exp(m\hat{\tau}_{\tilde{l}})s$$

and thus $\hat{s}_{\tilde{l}}$ is a parallel section of $d + m\eta_{\tilde{l}}$ and $s_{\tilde{l}}$ is a parallel section of $d + m\hat{\eta}_{\tilde{l}}$ where $\eta_{\tilde{l}} = \eta - d\tau_{\tilde{l}}$ and $\hat{\eta}_{\tilde{l}} = \hat{\eta} - d\hat{\tau}_{\tilde{l}}$. We therefore have the following proposition:

Proposition 5.26. *Suppose that $\hat{f}$ is a Darboux transform of f with parameter m and let l be any rank one subbundle of $f + \hat{f}$ with $l \cap s_0 = \{0\}$. Then there exist gauge potentials $\eta \in \Omega^1(f \wedge f^\perp)$ and $\hat{\eta} \in \Omega^1(\hat{f} \wedge \hat{f}^\perp)$ such that $s := f \cap l$ is a parallel subbundle of $d + m\hat{\eta}$ and $\hat{s} := \hat{f} \cap l$ is a parallel subbundle of $d + m\eta$.*

5.2.1 The enveloping sphere congruence

Suppose that $(s,t) = (4,2)$. In this subsection we will show that the nature of the enveloping sphere congruence s_0 determines when umbilics appear on a Darboux transform. Furthermore we will see how we can determine the middle connection of a Darboux transform. Suppose that f is an umbilic-free Lie applicable surface.

Proposition 5.27. *$\hat{f}$ has an umbilic at a point $p \in \Sigma$ if and only if s_0 coincides with one of the isothermic sphere congruences at p.*

Proof. Suppose that s_0 coincides with an isothermic sphere congruence $s \leq f$ at p. By Lemma 5.24, there exists a parallel section $\hat{\sigma} \in \Gamma\hat{f}$ of $d + m\eta$, where $\eta \in \Omega^1(s \wedge f^{\perp})$ is the isothermic gauge potential associated to s. Since s_0 coincides with s at p, we have that $\hat{\sigma}(p) \in s(p)^{\perp}$. Then

$$(d\hat{\sigma})_p = -m\eta_p\hat{\sigma}(p) \in s_0(p).$$

Therefore, $\hat{f}$ is umbilic at p.

Conversely, suppose that $\hat{f}$ is umbilic at p. Then there exists $\hat{s} \leq \hat{f}$ such that $(d\hat{\sigma})_p \in T_p\Sigma \otimes \hat{f}(p)$ for all $\hat{\sigma} \in \Gamma\hat{s}$. Since we assumed that s_0 is never a curvature sphere we have that $\hat{s} \cap s_0 = \{0\}$. Now we may choose $\eta \in \Omega^1(f \wedge f^{\perp})$ such that $\hat{s}$ is a parallel subbundle of $d + m\eta$. Let $\hat{\sigma} \in \Gamma\hat{s}$ be a parallel section of $d + m\eta$. Then at p

$$m\eta_p\hat{\sigma}(p) = -(d\hat{\sigma})_p \in T_p\Sigma \otimes \hat{f}(p).$$

Moreover, since $\eta \in \Omega^1(f \wedge f^{\perp})$, $\eta_p\hat{\sigma}(p)$ takes values in $f(p)^{\perp}$. Thus, $\eta_p\hat{\sigma}(p)$ takes values in $s_0 = \hat{f} \cap f^{\perp}$. Now for some complementary sphere congruence $s \leq f$ to s_0, we may write

$$\eta = \sigma_0 \wedge \omega_0 + \sigma \wedge \omega,$$

where $\omega_0, \omega \in \Omega^1(f^{\perp})$, $\sigma_0 \in \Gamma s_0$ and $\sigma \in \Gamma s$. Thus

$$\eta_p\hat{\sigma}(p) = (\sigma(p), \hat{\sigma}(p))\omega_p \bmod T_p^*\Sigma \otimes f.$$

Since s is complementary to s_0, we must have that $(\sigma(p), \hat{\sigma}(p))$ is non-zero and thus $\omega_p \in T_p\Sigma \otimes f$. Therefore, $\eta_p \in T_p\Sigma \otimes (s_0(p) \wedge f(p)^{\perp})$. Hence, by Lemma 4.18, s_0 coincides with an isothermic sphere congruence at p. $\square$

Now assume that f is an Ω-surface. Recall in Subsection 4.2.2 that we defined Δ_q and $\zeta_p \in \Gamma S^2 f^*$ associated to a Lie applicable surface. Using Proposition 4.20 we obtain the following corollary:

Corollary 5.28. *$\hat{f}$ is umbilic at p if and only if $\zeta_q(s_0(p), s_0(p)) = 0$.*

Now suppose that $\hat{f}$ is umbilic-free. Then by Corollary 5.28, $\zeta_q(s_0, s_0)$ is nowhere zero, i.e., $(\Delta_q\sigma_0) \cap s_0^{\perp} = \{0\}$ for any lift σ_0 of s_0. We may then define a sphere congruence, i.e., a subbundle of $\underline{\mathbb{R}}^{4,2}$ with signature $(3,1)$,

$$V_q := s_0 \oplus d\sigma_0(T\Sigma) \oplus \langle \Delta_q \sigma_0 \rangle.$$

Recall in Definition 2.40 that we defined the enveloping point s_∞ in the plane $f + \hat{f}$ of two Ribaucour transforms as the unique point in $f+\hat{f}$ satisfying $s_\infty^{(1)} \leq f + \hat{f}$. Taking lines between the corresponding curvature spheres of f and $\hat{f}$, we obtain s_∞ as the intersection of these two lines.

Proposition 5.29. *Let η^m denote the middle connection of f and $\hat{\eta}^m$ the middle connection of $\hat{f}$. Then $V_q^\perp = s \oplus \hat{s}$ where $s \leq f$ is a parallel subbundle of $d + m\eta^m$ and $\hat{s} \leq \hat{f}$ is a parallel subbundle of $d + m\eta^m$. Furthermore, $s_\infty \leq V_q^\perp$.*

To prove Proposition 5.29 we shall use the following lemma:

Lemma 5.30. *Suppose that η^m is the middle connection. Let $\tau \in \Gamma(\wedge^2 f)$. Then,*

$$\eta^m(X) d_X \tau - \epsilon^2 \eta^m(Y) d_Y \tau = -\epsilon^2 \tau,$$

where $X \in \Gamma T_1$, $Y \in \Gamma T_2$ such that

$$q(X,X) = -\epsilon^2 \quad \textit{and} \quad q(Y,Y) = 1.$$

Proof. Let $\sigma_1 \in \Gamma s_1$, $\sigma_2 \in \Gamma s_2$ be the special lifts of the curvature spheres such that

$$q = -\epsilon^2 (d\sigma_2, d\sigma_2) + (d\sigma_1, d\sigma_1)$$

and let $\tau = \sigma_1 \wedge \sigma_2 \in \Gamma \wedge^2 f$. Recall from Remark 4.14 that the middle connection is given by

$$\eta^m = \sigma_1 \wedge \star d\sigma_1 + \epsilon^2 \sigma_2 \wedge \star d\sigma_2.$$

Thus, for $v \in \Gamma \underline{\mathbb{R}}^{4,2}$,

$$\begin{aligned} (\eta^m(X) d_X \tau) v &= (\epsilon^2 \sigma_2 \wedge d_X \sigma_2)(\sigma_1 \wedge d_X \sigma_2) v \\ &= -\epsilon^2 (d_X \sigma_2, d_X \sigma_2)(\sigma_1, v)\sigma_2 \\ &= -\epsilon^2 (\sigma_1, v)\sigma_2. \end{aligned}$$

Similarly, $(\eta^m(Y) d_Y \tau) v = -(\sigma_2, v)\sigma_1$. Hence,

$$(\eta^m(X) d_X \tau - \epsilon^2 \eta^m(Y) d_Y \tau) v = -\epsilon^2 (\sigma_1, v)\sigma_2 + \epsilon^2 (\sigma_2, v)\sigma_1 = -\epsilon^2 \tau v$$

and the result follows. ∎

Proof of Proposition 5.29. Let $\hat{\sigma} \in \Gamma\hat{s}$ be a parallel section of $d + m\eta^m$ and let $\sigma_0 \in \Gamma s_0$. Then,

$$\begin{aligned}
(\Delta_q\sigma_0, \hat{\sigma}) &= (d_X d_X \sigma_0 - \epsilon^2 d_Y d_Y \sigma_0, \hat{\sigma}) \\
&= -(d_X\sigma_0, d_X\hat{\sigma}) + \epsilon^2(d_Y\sigma_0, d_Y\hat{\sigma}) \\
&= m((d_X\sigma_0, \eta^m(X)\hat{\sigma}) - \epsilon^2(d_Y\sigma_0, \eta^m(Y)\hat{\sigma})) \\
&= -m(\eta^m(X)d_X\sigma_0 - \epsilon^2\eta^m(Y)d_Y\sigma_0, \hat{\sigma}).
\end{aligned}$$

Now, there exists $\tau \in \Gamma(\wedge^2 f)$ such that $\sigma_0 = \tau\hat{\sigma}$. Hence,

$$(\Delta_q\sigma_0, \hat{\sigma}) = -m((\eta^m(X)d_X\tau - \epsilon^2\eta^m(Y)d_Y\tau)\hat{\sigma}, \hat{\sigma}) = m\epsilon^2(\tau\hat{\sigma}, \hat{\sigma}),$$

by Lemma 5.30. By the skew-symmetry of τ, $(\Delta_q\sigma_0, \hat{\sigma})$ vanishes.

By a symmetric argument, $(\Delta_q\sigma_0, \sigma)$ vanishes, where σ is a parallel section of $d + m\hat{\eta}^m$.

Now let $\sigma_\infty \in \Gamma s_\infty$. Then $d\sigma_\infty \in \Omega^1(f + \hat{f})$. Then using that $d\sigma_0 \in \Omega^1((f + \hat{f})^\perp)$ for any $\sigma_0 \in \Gamma s_0$, it is clear that $s_\infty \leq (s_0 \oplus d\sigma_0(T\Sigma))^\perp$ and one can easily check that $(\Delta_q\sigma_0, \sigma_\infty)$ vanishes. □

Remark 5.31. *Proposition 5.29 tells us how to obtain the middle connection of $\hat{f}$: let $\hat{s} \leq \hat{f}$ be the parallel subbundle of $d + m\eta^m$ and let $s := f \cap l$, where l is the line spanned by $\hat{s}$ and s_∞. Then by Proposition 5.22*

$$d + t\hat{\eta}^m = \Gamma_s^{\hat{s}}(1 - t/m) \cdot (d + t\eta^m).$$

5.2.2 Isothermic sphere congruences

Now suppose that we are working with the isothermic connection η^+ associated to the isothermic sphere congruence s^+. Then if $\hat{s} \leq \hat{f}$ is the parallel subbundle of $d + m\eta^+$ then by Proposition 5.22, $\hat{\eta}$ defined by

$$d + t\hat{\eta} = \Gamma_{s^+}^{\hat{s}}(1 - t/m) \cdot (d + t\eta^+)$$

is a closed one-form. Recall that we split $\eta = \eta_0 + \eta_{s^+}$ and $\hat{\eta} = \hat{\eta}_0 + \hat{\eta}_{\hat{s}}$, where $\eta_0, \hat{\eta}_0 \in \Omega^1(s_0 \wedge l^\perp)$, $\eta_{s^+} \in \Omega^1(s^+ \wedge l^\perp)$ and $\hat{\eta}_{\hat{s}} \in \Omega^1(\hat{s} \wedge l^\perp)$. Now, in the proof of Proposition 5.22 we saw that $\eta_0 = \hat{\eta}_0$ and since we are working with the isothermic connection $\eta^+ \in \Omega^1(s^+ \wedge f^\perp)$, we have that $\eta_0 = 0$. Thus, $\hat{\eta} \in \Omega^1(\hat{s} \wedge l^\perp)$. Hence, $\hat{s}$ is isothermic and we shall denote it $\hat{s}^+$. A symmetric argument yields an analogous result for s^-.

Proposition 5.32. *We may label the isothermic sphere congruences $\hat{s}^+$ and $\hat{s}^-$ of $\hat{f}$ such that $\hat{s}^\pm$ is a parallel subbundle of $d + m\eta^\pm$. Furthermore*

$$d + t\hat{\eta}^\pm = \Gamma_{s^\pm}^{\hat{s}^\pm}(1 - t/m) \cdot (d + t\eta^\pm).$$

Remark 5.33. *Proposition 5.32 shows that Darboux transforms of Lie applicable surfaces are* *induced by the Darboux transforms of their isothermic sphere congruences [11, 50]. This is our justification for using the term "Darboux transform" instead of "Bäcklund transform".*

We now give a result concerning the lines joining "opposite" isothermic sphere congruences:

Proposition 5.34. *Let $l_1 = s^+ \oplus \hat{s}^-$ and $l_2 = s^- \oplus \hat{s}^+$. Then $l_1 \cap l_2 = s_\infty$.*

Proof. By Proposition 4.22, $\eta^- = \sigma^- \wedge d\sigma^+$, where $\sigma^\pm$ are Christoffel dual lifts of $s^\pm$. By Proposition 5.32, there exists $\hat{\sigma} \in \Gamma\hat{s}^-$ such that $\hat{\sigma}$ is a parallel section of $d + m\eta^-$. Thus,

$$d\hat{\sigma} = -m(\sigma^-, \hat{\sigma})d\sigma^+ \, mod \, \Omega^1(f).$$

Hence, $\sigma_\infty := \hat{\sigma} + m(\sigma^-, \hat{\sigma})\sigma^+ \in \Gamma l_1$ and satisfies $d\sigma_\infty \in \Omega^1(f + \hat{f})$. Since s_∞ is the unique point in $f + \hat{f}$ satisfying $s_\infty^{(1)} \leq f + \hat{f}$, we have that $\sigma_\infty \in \Gamma s_\infty$. Therefore, $s_\infty \leq l_1$. Similarly, $s_\infty \leq l_2$ and the result follows. $\square$

Further work

As shown in Section 4.3, given an Ω-surface we can determine a parallel family of associate Ω-surfaces. This is yet another transformation of Ω-surfaces. It would be useful to have a relation between the middle connections of these surfaces akin to the relation given in [11, 18, 64] between the connections of isothermic surfaces and their Christoffel duals.

Chapter 6

Polynomial conserved quantities of Ω-surfaces

In [18, 64], based on an idea[1] of Burstall and Calderbank, isothermic surfaces admitting polynomial conserved quantities are considered. Isothermic surfaces admitting degree d polynomial conserved quantities are defined to be special isothermic surfaces of type d. It is shown that type 1 special isothermic surfaces coincide with surfaces of constant mean curvature in certain space forms and those of type 2 are the classical special isothermic surfaces of Darboux [33] in certain space forms.

We shall take a similar approach and consider Ω-surfaces that admit polynomial conserved quantities. The situation is somewhat less clear as we have a gauge orbit of connections to consider - this will be resolved by using the middle connection. We will see that

- Isothermic surfaces, Guichard surfaces and L-isothermic surfaces can be characterised as Ω-surfaces admitting certain linear conserved quantities.
- The special Ω-surfaces of Eisenhart [39] in space forms are Ω-surfaces admitting degree 2 conserved quantities.
- Special Guichard surfaces [37] and special isothermic surfaces in space forms are Ω-surfaces admitting a linear conserved quantity and a quadratic conserved quantity.

Furthermore we will study the effect of the transformations of Chapter 5 on such surfaces. For example we will show that the Eisenhart transformation for Guichard surfaces coincides with the Darboux transformations that preserve a certain linear conserved quantity.

6.1 Polynomial conserved quantities

Suppose that $f : \Sigma \to \mathcal{Z}$ is a Lie applicable surface with family of flat connections $d^t = d + t\eta$. We now give a definition that is analogous to that of [18, 64]:

[1] This idea originates from the notion of polynomial killing field introduced in [4].

Definition 6.1. *A non-zero polynomial $p \in \Gamma\underline{\mathbb{R}}^{4,2}[t]$ is called a polynomial conserved quantity of d^t if $p(t)$ is a parallel section of d^t for all $t \in \mathbb{R}$.*

The following lemma shows that the existence of polynomial conserved quantities is gauge invariant. Suppose that $\tilde{\eta}$ is in the gauge orbit of η so that $\tilde{\eta} = \eta - d\tau$ for $\tau \in \Gamma(\wedge^2 f)$.

Lemma 6.2. *Suppose that p is a polynomial conserved quantity of $d + t\eta$. Then $\tilde{p}(t) = \exp(t\tau)p(t)$ is a polynomial conserved quantity of $d + t\tilde{\eta}$ with $\tilde{p}(0) = p(0)$.*

Proof. This follows immediately from (5.1). □

Using an identical argument to [64, Proposition 2.12], one obtains the following lemma:

Lemma 6.3. *Suppose that p is a polynomial conserved quantity of d^t. Then the polynomial $(p(t), p(t)) \in \mathbb{R}[t]$ has constant coefficients.*

From now on we shall assume that f is an umbilic-free Ω-surface.

Proposition 6.4. *Suppose that $p(t) = \sum_{k=0}^{d} p_k t^k$ is a degree d polynomial conserved quantity of the middle connection $d + t\eta^m$ of f. Then*

1. *p_0 is constant.*
2. *p_d is a section of f.*
3. *For any $\tau \in \Gamma(\wedge^2 f)$, $\tilde{p}(t) = \exp(t\tau)p(t)$ has degree at most d and the coefficient of t^d is given by $p_d + \tau p_{d-1}$.*

Proof. Consider the polynomial $(d + t\eta^m)p(t)$ whose coefficients take values in $\Omega^1(\underline{\mathbb{R}}^{4,2})$:

$$0 = (d + t\eta^m)p(t) = dp_0 + \sum_{k=1}^{d} t^k(dp_k + \eta^m p_{k-1}) + t^{d+1}\eta^m p_d.$$

Therefore $dp_0 = 0$ and thus p_0 is constant. Furthermore, $\eta^m p_d = 0$. Now by Remark 4.14, in terms of special lifts of the curvature spheres, the middle connection is given by

$$\eta^m = \sigma_1 \wedge \star d\sigma_1 + \epsilon^2 \sigma_2 \wedge \star d\sigma_2.$$

Therefore,

$$0 = \eta^m p_d = (\sigma_1, p_d) \star d\sigma_1 - (\star d\sigma_1, p_d)\sigma_1 + \epsilon^2(\sigma_2, p_d) \star d\sigma_2 - (\star d\sigma_2, p_d)\sigma_2. \tag{6.1}$$

Since $d_Y\sigma_1$ and $d_X\sigma_2$ never belong to f for any non-zero $X \in \Gamma T_1$, $Y \in \Gamma T_2$, we have that

$$(\sigma_1, p_d) = (\sigma_2, p_d) = 0.$$

Thus, $p_d \in \Gamma f^{\perp}$. It then follows from (6.1) and the linear independence of σ_1 and σ_2 that

$$(\star d\sigma_1, p_d) = (\star d\sigma_2, p_d) = 0.$$

Therefore, since $f^{(1)} = f^{\perp}$, p_d takes values in $(f^{\perp})^{\perp}$. This implies that $p_d \in \Gamma f$.

Since $p_d \in \Gamma f$, $\tau p_d = 0$ for any $\tau \in \Gamma(\wedge^2 f)$. Therefore,

$$\exp(t\tau)p(t) = p(t) + t\tau p(t) = p_0 + \sum_{k=1}^{d} t^k (p_k + \tau p_{k-1}) + t^{d+1}\tau p_d$$

is a polynomial of degree at most d and the coefficient of t^d is $p_d + \tau p_{d-1}$. □

Remark 6.5. *For polynomial conserved quantities of Ω_0-surfaces, 2 and 3 of Proposition 6.4 do not necessarily hold. We shall not consider general polynomial conserved quantities of Ω_0-surfaces, however in Subsection 7.3.2 we shall consider constant conserved quantities.*

More in fact can be said about the top term of a polynomial conserved quantity of the middle connection, but first we will investigate polynomial conserved quantities of isothermic connections. Suppose that $s : \Sigma \to \mathbb{P}(\mathcal{L})$ is an isothermic sphere congruence with isothermic connection $d + t\eta^s$ and suppose that s defines a conformal structure on $T\Sigma$. We may then define the central sphere congruence of s:

$$V^s := s^{(1)} \oplus \langle \Delta\sigma \rangle,$$

where $\Delta\sigma$ is the Laplacian of any lift $\sigma \in \Gamma s$ with respect to the conformal structure induced by s. By applying the arguments of [64, Proposition 2.12] to $\mathbb{R}^{4,2}$ we obtain the following proposition:

Proposition 6.6. *Suppose that p is a degree d polynomial conserved quantity of $d + t\eta^s$. Then p_d is a parallel section of $(V^s)^{\perp}$ with respect to the connection on $(V^s)^{\perp}$ induced by the splitting $\underline{\mathbb{R}}^{4,2} = V^s \oplus (V^s)^{\perp}$.*

Recall from Proposition 4.22 that in terms of the Christoffel dual lifts $\sigma^{\pm}$ of the isothermic sphere congruences of f we have that

$$d + t\eta^{\pm} = \exp(t\tau^{\pm}) \cdot (d + t\eta^m),$$

where $\tau = \pm\frac{1}{2}\sigma^+ \wedge \sigma^-$.

Corollary 6.7. *Suppose that p is a degree d polynomial conserved quantity of the middle connection of f. Then $p_d = (\sigma^+ \odot \sigma^-)p_{d-1}$. Furthermore, $(\sigma^{\pm}, p_{d-1})$ are constants.*

Proof. By Proposition 6.4 the top term of p lies in f. Thus $p_d = \lambda\sigma^+ + \mu\sigma^-$ for some smooth functions λ and μ. By Lemma 6.2, $p^+(t) = \exp(\tau^+)p(t)$ is a polynomial conserved quantity of $d + t\eta^+$ and, by Proposition 6.4, p^+ has degree at most d and $p_d + \tau^+ p_{d-1}$ is the coefficient of t^d. Now

$$p_d + \tau^+ p_{d-1} = \lambda\sigma^+ + \mu\sigma^- + \frac{1}{2}((\sigma^+, p_{d-1})\sigma^- - (\sigma^-, p_{d-1})\sigma^+). \tag{6.2}$$

By Proposition 6.6, $p_d + \tau^+ p_{d-1}$ is a parallel section of $f \cap (V^+)^\perp$. Then, by Remark 4.23, $p_d + \tau^+ p_{d-1} = \alpha\sigma^-$, for some constant α. Comparing this with (6.2), we have that $\lambda = \frac{1}{2}(\sigma^-, p_{d-1})$ and that $\mu + \frac{1}{2}(\sigma^+, p_{d-1})$ is constant. Similarly, one can show that $\mu = \frac{1}{2}(\sigma^+, p_{d-1})$ and $\lambda + \frac{1}{2}(\sigma^-, p_{d-1})$ is constant. Therefore,

$$p_d = \frac{1}{2}((\sigma^-, p_{d-1})\sigma^+ + (\sigma^+, p_{d-1})\sigma^-) = (\sigma^+ \odot \sigma^-)p_{d-1},$$

with $(\sigma^\pm, p_{d-1})$ constant. □

Corollary 6.8. *Suppose that p is a polynomial conserved quantity of degree d of the middle connection of f. Then for $\tau \in \Gamma(\wedge^2 f)$, $\tilde{p}(t) := \exp(t\tau)p(t)$ has degree strictly less than d if and only if $p_{d-1} \in \Gamma(s^+)^\perp$ (or $p_{d-1} \in \Gamma(s^-)^\perp$) and $\tau = \tau^+$ (respectively, $\tau = \tau^-$).*

Proof. From Proposition 6.4, we have that the coefficient of t^d of $\tilde{p}$ is given by $p_d + \tau p_{d-1}$. We may write $\tau = \beta\sigma^+ \wedge \sigma^-$, where β is a smooth function and $\sigma^\pm$ are Christoffel dual lifts. Then by Corollary 6.7, we have that

$$p_d + \tau p_{d-1} = \tfrac{1}{2}((\sigma^-, p_{d-1})\sigma^+ + (\sigma^+, p_{d-1})\sigma^-) + \beta((\sigma^+, p_{d-1})\sigma^- - (\sigma^-, p_{d-1})\sigma^+).$$

Therefore, the t^d coefficient of $\tilde{p}$ vanishes if and only if

$$(\beta + \tfrac{1}{2})(\sigma^+, p_{d-1}) = 0 = (\beta - \tfrac{1}{2})(\sigma^-, p_{d-1}). \tag{6.3}$$

Since the top term of p is given by $(\sigma^+ \odot \sigma^-)p_{d-1}$, we cannot have that (σ^+, p_{d-1}) and (σ^-, p_{d-1}) both vanish as this would imply that p has degree strictly less than d. Therefore, without loss of generality, assume that $(\sigma^-, p_{d-1}) \neq 0$. Then (6.3) is equivalent to $\beta = \frac{1}{2}$ and $(\sigma^+, p_{d-1}) = 0$, i.e., $\tau = \frac{1}{2}\sigma^+ \wedge \sigma^- = \tau^+$ and $p_{d-1} \in \Gamma(s^+)^\perp$. □

Corollary 6.9. *Suppose that p is a polynomial conserved quantity of the middle connection. Then the degree d of p is invariant under gauge transformation if and only if $(p(t), p(t))$ is a polynomial of degree $2d - 1$.*

Proof. By Proposition 6.4 we have that $p_d \in \Gamma f$. Therefore there is no $2d$-term of $(p(t), p(t))$. Now the coefficient of t^{2d-1} in $(p(t), p(t))$ is $2(p_d, p_{d-1})$ and by Corollary 6.7,

$$(p_d, p_{d-1}) = (\sigma^+, p_{d-1})(\sigma^-, p_{d-1}).$$

Therefore, by Corollary 6.8, the coefficient of t^{2d-1} vanishes if and only if there exists $\tau \in \Gamma(\wedge^2 f)$ such that $\exp(t\tau)p(t)$ has degree strictly less than d. □

Analogously to [18, 64], we make the following definition:

Definition 6.10. *An umbilic-free Ω-surface is a special Ω-surface of type d if its middle connection admits a non-zero polynomial conserved quantity of degree d.*

Remark 6.11. *Type zero special Ω-surfaces do not exist as this would imply that there exists $\mathfrak{q} \in (\mathbb{R}^{4,2})^\times$ such that $\mathfrak{q} \in \Gamma f$, implying that f is totally umbilic.*

Now suppose that f is a special Ω-surface of type d with degree d conserved quantity p. Let m be a non-zero root of the polynomial $(p(t), p(t))$. Then $p(m)$ is lightlike and is a parallel section of $d + m\eta^m$. If we let $s_0 := f \cap \langle p(m)\rangle^\perp$ and define $\hat{f} := s_0 \oplus \langle p(m)\rangle$, then $\hat{f}$ is a Darboux transform of f with parameter m. Unsurprisingly, [18, 64] lead us to make the following definition:

Definition 6.12. *The Darboux transforms $\hat{f}$ of f such that $p(m) \in \Gamma\hat{f}$ for some $m \in \mathbb{R}^\times$ are called the complementary surfaces of f with respect to p.*

Remark 6.13. *Since the degree of $(p(t), p(t))$ is less than or equal to $2d-1$, we have at most $2d-1$ complementary surfaces.*

6.2 Transformations of polynomial conserved quantities

We would now like to investigate how polynomial conserved quantities behave when we apply the transformations of Chapter 5. Suppose that f is a special Ω-surface of type d and let p be associated degree d polynomial conserved quantity of the middle connection.

6.2.1 Calapso transformations

Suppose that $f^t := T(t)f$ is a Calapso transform of f, where $T(t)$ denotes the local trivialising orthogonal gauge transformations of the middle connection $d + t\eta^m$ of f.

Proposition 6.14. *The middle connection of f^t admits a degree d polynomial conserved quantity p^t defined by*

$$p^t(s) := T(t)p(s+t)$$

with constant term $p^t(0) = T(t)p(t)$.

Proof. By Proposition 5.8, the middle connection of f^t is given by

$$d + s(\eta^t)^m = T(t) \cdot (d + (s+t)\eta^m).$$

Then it follows immediately that p^t is a polynomial conserved quantity of the middle connection of f^t. Furthermore, the coefficient of s^d in $p^t(s)$ is $T(t)p_d \neq 0$. Hence p^t has degree d. □

We have thus proved the following Theorem:

Theorem 6.15. *The Calapso transforms of special Ω-surfaces of type d are special Ω-surfaces of type d.*

6.2.2 Darboux transformations

Suppose that $\hat{f}$ is an umbilic-free Darboux transform of f with parameter $m \in \mathbb{R}^\times$. Then by Remark 5.31, the middle connection of $\hat{f}$ is given by

$$d + m\hat{\eta}^m = \Gamma_s^{\hat{s}}(1 - t/m) \cdot (d + t\eta^m)$$

where $\hat{s} \leq \hat{f}$ is the parallel subbundle of $d + m\eta^m$ and $s \leq f$ is the parallel subbundle of $d + m\hat{\eta}^m$. Therefore, $\Gamma_s^{\hat{s}}(1 - t/m)p(t)$ is a conserved quantity of $d + t\hat{\eta}^m$. Using the splitting

$$\underline{\mathbb{R}}^{4,2} = s \oplus \hat{s} \oplus (s \oplus \hat{s})^\perp,$$

we shall write $p(t)$ as

$$p(t) = [p(t)]_s + [p(t)]_{\hat{s}} + [p(t)]_{(s\oplus\hat{s})^\perp}.$$

Thus

$$\Gamma_s^{\hat{s}}(1 - t/m)p(t) = \tfrac{m}{m-t}[p(t)]_s + \tfrac{m-t}{m}[p(t)]_{\hat{s}} + [p(t)]_{(s\oplus\hat{s})^\perp}.$$

Proposition 6.16. *$\hat{p}(t) := (1-t/m)\Gamma_s^{\hat{s}}(1-t/m)p(t)$ defines a degree $d+1$ polynomial conserved quantity of $d + t\hat{\eta}^m$. Furthermore, if $p(m) \in \Gamma\hat{s}^\perp$ then $\hat{p}(t) := \Gamma_s^{\hat{s}}(1 - t/m)p(t)$ is a degree d polynomial conserved quantity with $(\hat{p}(t), \hat{p}(t)) = (p(t), p(t))$. In either case $\hat{p}(0) = p(0)$.*

Proof. First note that by Proposition 6.4, the top term p_d of $p(t)$ lies in f. Therefore, $[p(t)]_{\hat{s}}$ has degree strictly less than d. Hence,

$$(1 - t/m)\Gamma_s^{\hat{s}}(1 - t/m)p(t) = [p(t)]_s + t\frac{(m-t)^2}{m^2}[p(t)]_{\hat{s}} + \frac{m-t}{m}[p(t)]_{(s\oplus\hat{s})^\perp}$$

is a polynomial conserved quantity of degree $d+1$ of $d + t\hat{\eta}_m$.

Now let $\sigma \in \Gamma s$ and $\hat{\sigma} \in \Gamma\hat{s}$ such that $(\sigma, \hat{\sigma}) = -1$. Then $[p(t)]_s = -(p(t), \hat{\sigma})\sigma$. Therefore, if $p(m) \in \Gamma\hat{s}^\perp$, then $[p(t)]_s$ has a root at m and $\frac{m}{m-t}[p(t)]_s$ is a polynomial of degree less than d. Therefore, $\hat{p}(t) = \Gamma_s^{\hat{s}}(1 - t/m)p(t)$ is a degree d polynomial conserved quantity of $d + t\hat{\eta}^m$. Furthermore, since $\Gamma_s^{\hat{s}}(1 - t/m)$ takes values in $\mathrm{O}(4, 2)$ for all t, $(\hat{p}(t), \hat{p}(t)) = (p(t), p(t))$.

Finally, we have that in either case $\hat{p}(0) = p(0)$ because $\Gamma_s^{\hat{s}}(1)$ is the identity. $\square$

We have thus proved the following theorem:

Theorem 6.17. *$\hat{f}$ is a special Ω-surface of type $d+1$. Furthermore, if $p(m) \in \Gamma\hat{s}^\perp$, then $\hat{f}$ is a special Ω-surface of type d.*

Remark 6.18. *Since $\{d^t = d + t\eta^m\}_{t\in\mathbb{R}}$ is a family of metric connections, we have that*

$$d(p(m), \hat{\sigma}) = (d^m p(m), \hat{\sigma}) + (p(m), d^m\hat{\sigma}) = 0,$$

where $\hat{\sigma} \in \Gamma\hat{s}$ is a parallel section of d^m. Therefore, if $p(m) \in \Gamma\hat{s}^\perp$ at a point $p \in \Sigma$, then $p(m) \in \Gamma\hat{s}^\perp$ throughout Σ.

6.3 Type one special Ω-surfaces

In this section we shall see that special Ω-surfaces of type one, i.e., Ω-surfaces whose middle connection admits a linear conserved quantity $p(t)$, include isothermic surfaces, Guichard surfaces and L-isothermic surfaces. Furthermore the familiar transformations of these surfaces are restrictions of the transformations of Chapter 5. For example we shall show that the Eisenhart transformations for Guichard surfaces are Darboux transformations preserving the linear conserved quantity.

Suppose that f is a special Ω-surface of type one and let $p(t) = p_0 + tp_1$ be the associated linear conserved quantity of the middle connection. By Proposition 6.4, p_0 is constant and $p_1 \in \Gamma f$. From Corollary 6.7 we immediately obtain the following lemma:

Lemma 6.19. *For Christoffel dual lifts $\sigma^{\pm}$, $(\sigma^{\pm}, p_0)$ are constant and*

$$p(t) = \exp(t\,\sigma^{+} \odot \sigma^{-})p_0.$$

Using Lemma 2.23 we obtain the following corollary:

Corollary 6.20. *f lies nowhere in $\langle p_0 \rangle^{\perp}$.*

Proof. Suppose that at a point $p \in \Sigma$, $f(p) \leq \langle p_0 \rangle^{\perp}$. Then $(\sigma^{\pm}(p), p_0) = 0$ and by Lemma 6.19, $(\sigma^{\pm}, p_0) = 0$ throughout Σ. Therefore, since $\sigma^{\pm}$ span f, $f \leq \langle p_0 \rangle^{\perp}$. Then by Lemma 2.23, this contradicts f being an umbilic-free Legendre map. $\square$

6.3.1 Isothermic surfaces

Suppose that $\mathfrak{p} \in \mathbb{R}^{4,2}$ is a point sphere complex. Then $\langle \mathfrak{p} \rangle^{\perp}$ is a (Riemannian or Lorentzian) conformal subgeometry of $\mathbb{R}^{4,2}$. Let $\mathcal{L}^{\mathfrak{p}}$ denote the lightcone of $\langle \mathfrak{p} \rangle^{\perp}$. In [11, 18, 50, 64], isothermic surfaces are characterised as the surfaces $\Lambda : \Sigma \to \mathbb{P}(\mathcal{L}^{\mathfrak{p}})$ that admit a non-zero closed one-form

$$\eta \in \Omega^1(\Lambda \wedge \Lambda^{(1)}).$$

Let $f : \Sigma \to \mathcal{Z}$ be the Legendre lift of Λ. Then $\Lambda = f \cap \langle \mathfrak{p} \rangle^{\perp}$ and η takes values in $f \wedge f^{\perp}$. Furthermore, the quadratic differential

$$q(X, Y) = tr(\sigma \to \eta(X) d_Y \sigma)$$

coincides with the holomorphic[2] (with respect to the conformal structure induced by Λ) quadratic differential defined in [18, 64]. Thus, q is non-degenerate and f is an Ω-surface. Furthermore,

$$(d + t\eta)\mathfrak{p} = 0,$$

[2] That is, locally there exists a holomorphic coordinate z on Σ such that $q^{2,0} := q(\frac{\partial}{\partial z}, \frac{\partial}{\partial z})dz^2 = dz^2$ and $I = e^{2u} dz d\bar{z}$.

i.e., $\mathfrak{p}$ is a constant conserved quantity of $d + t\eta$. Thus, if $\tau \in \Gamma(\wedge^2 f)$ such that the middle connection is given by

$$d + t\eta^m = \exp(t\tau) \cdot (d + t\eta),$$

then we have that $p(t) = \exp(t\tau)\mathfrak{p}$ is a linear conserved quantity of $d + t\eta^m$. Moreover, $(p(t), p(t)) = (\mathfrak{p}, \mathfrak{p})$ is a non-zero constant.

Conversely, suppose that f is a special Ω-surface of type one with linear conserved quantity p and suppose that $(p(t), p(t))$ is a non-zero constant. If we let $\mathfrak{p} := p_0$, then $\mathfrak{p}$ is a point sphere complex and $\langle \mathfrak{p} \rangle^\perp$ defines a (Riemannian or Lorentzian) conformal geometry. By Corollaries 6.8 and 6.9, we have that one of the isothermic sphere congruences, without loss of generality $\Lambda := s^+$, of f takes values in $\langle \mathfrak{p} \rangle^\perp$. Then Λ is an isothermic surface and

$$\eta^+ \in \Omega^1(\Lambda \wedge \Lambda^{(1)})$$

is its associated closed one form. We have therefore arrived at the following theorem:

Theorem 6.21. *Special Ω-surfaces of type one whose degree one polynomial conserved quantity p satisfies $(p(t), p(t))$ being a non-zero constant are the isothermic surfaces of the conformal geometry defined by $\langle p(0) \rangle^\perp$.*

We shall now see how the classical transformations of isothermic surfaces are induced by the transformations of Chapter 5: suppose that f is an umbilic-free Ω-surface such that $\Lambda := s^+$ is an isothermic surface in $\langle \mathfrak{p} \rangle^\perp$. Then

$$p(t) := \exp(t\tau)\mathfrak{p}$$

is a polynomial conserved quantity of the middle connection, where $\tau = \frac{1}{2}\sigma^+ \wedge \sigma^-$ for Christoffel dual lifts $\sigma^\pm$.

Calapso transforms

Suppose that $f^t = T(t)f$ is a Calapso transform of f. Then by Lemma 5.4, we may assume that $T(t)$ is the gauge transformation of $d + t\eta^+$. Now

$$d(T(t)\mathfrak{p}) = T(t)(d + t\eta^+)\mathfrak{p} = 0.$$

Thus, $T(t)\mathfrak{p}$ is constant and, by premultiplying by an appropriate Lie sphere transformation, we may assume that it is $\mathfrak{p}$. Then $T(t)\Lambda \leq f^t$ is a Calapso transform of the isothermic surface Λ in the sense of [18, 50, 64].

Theorem 6.22. *The Calapso transforms of f are the Legendre lifts of the Calapso transforms of Λ.*

Darboux transforms

Suppose that $\hat{f}$ is an umbilic-free Darboux transform of f with parameter $m \in \mathbb{R}^\times$. Assume that the parallel subbundle $\hat{s} \leq \hat{f}$ of $d + m\eta^m$ satisfies $\hat{s} \leq \langle p(m)\rangle^\perp$. Then $\hat{s}^+ := \exp(m\tau)\hat{s}$ is a parallel subbundle of $d + m\eta^+$ and

$$(\hat{s}^+, \mathfrak{p}) = (\exp(m\tau)\hat{s}, \exp(m\tau)p(m)) = (\hat{s}, p(m)) = 0.$$

Thus, $\hat{s}^+$ is a Darboux transform in the sense of [50, 64, 18, 11] of the isothermic surface s^+.

Conversely, if $\hat{\Lambda}$ is a Darboux transform of Λ with parameter m then $\hat{s} := \exp(-m\tau)\hat{\Lambda}$ is a parallel subbundle of $d + m\eta^m$ and

$$(\hat{s}, p(m)) = (\exp(-m\tau)\hat{\Lambda}, \exp(-m\tau)\mathfrak{p}) = 0.$$

We have therefore arrived at the following theorem:

Theorem 6.23. *The umbilic-free Darboux transforms $\hat{f}$ of f with parameter m such that $p(m) \in \Gamma\hat{s}^\perp$ are the Legendre lifts of Darboux transforms of Λ with parameter m.*

The Christoffel transformation

Now suppose that $\mathfrak{p}$ satisfies $|\mathfrak{p}|^2 = -1$ and let $\mathfrak{q}_\infty \in \langle\mathfrak{p}\rangle^\perp$ be a null space form vector. Then $\mathfrak{Q}^3$ is isometric to a Euclidean geometry. Let $\mathfrak{f} : \Sigma \to \mathfrak{Q}^3$ denote the corresponding space form projection of f and let $\nu : \Sigma \to \mathbb{R}^3$ be a corresponding surface in Euclidean space. Then

$$\eta^+ = \mathfrak{f} \wedge d\mathfrak{f} \circ A$$

for some $A \in \Gamma End(T\Sigma)$ and $(\eta^+\mathfrak{p}, \mathfrak{q}_\infty) = 0$. By comparing this with Section 4.3, we have that there is an associate surface ν^D of ν such that

$$\frac{1}{\kappa_1\kappa_2^D} + \frac{1}{\kappa_2\kappa_1^D} = 0.$$

Now using Rodrigues' equations one can deduce that in terms of curvature line coordinates (u, v) for ν (and thus ν^D) we have that

$$\frac{(\nu_u, \nu_u)}{(\nu_v, \nu_v)} = \frac{\kappa_2^2(\kappa_1^D)^2}{\kappa_1^2(\kappa_2^D)^2}\frac{(\nu_u^D, \nu_u^D)}{(\nu_v^D, \nu_v^D)} = \frac{(\nu_u^D, \nu_u^D)}{(\nu_v^D, \nu_v^D)}.$$

Hence, the conformal structures induced by ν and ν^D are equivalent. Therefore, since ν and ν^D have parallel curvature directions and induce the same conformal structure, they are Christoffel transforms of each other.

6.3.2 Guichard surfaces

Suppose that f is a special Ω-surface of type one admitting a polynomial conserved quantity $p(t)$ such that $(p(t), p(t))$ is a linear polynomial with non-zero constant term. Then $\mathfrak{p} := p_0$ defines a point sphere complex for the conformal geometry $\langle \mathfrak{p} \rangle^{\perp}$. We will show that f satisfies Calapso's equation [20]

$$EG(\kappa_1 - \kappa_2)^2 = E - \epsilon^2 G$$

in space forms with point sphere complex $\mathfrak{p}$, for a suitable choice of curvature line coordinates.

Let $\mathfrak{q} \in \langle \mathfrak{p} \rangle^{\perp}$ be a space form vector and let $\mathfrak{f} : \Sigma \to \mathfrak{Q}^3$ denote the projection of f into the space form $\mathfrak{Q}^3$ defined by $\mathfrak{q}$ and $\mathfrak{p}$ and let $\mathfrak{t} : \Sigma \to \mathfrak{P}^3$ be the corresponding tangent plane congruence. Since the polynomial $(p(t), p(t))$ has constant coefficients, we have that $(\mathfrak{p}, p_1)$ is constant. By adjusting the parameter t, we may assume that $(\mathfrak{p}, p_1) = 1$. Let $\tau := (p_1, \mathfrak{q}) p_1 \wedge \mathfrak{f} \in \Gamma(\wedge^2 f)$. Then by Lemma 6.2,

$$\tilde{p}(t) := \exp(t\tau) p(t) = \mathfrak{p} + t((p_1, \mathfrak{q})\mathfrak{f} + p_1)$$

is a linear conserved quantity of $d + t\tilde{\eta} = \exp(t\tau) \cdot (d + t\eta^m)$. Now $\tilde{p}_1 := (p_1, \mathfrak{q})\mathfrak{f} + p_1$ satisfies $(\tilde{p}_1, \mathfrak{p}) = 1$ and $(\tilde{p}_1, \mathfrak{q}) = 0$. Thus, $\tilde{p}_1 = -\mathfrak{t}$. Therefore, $-d\mathfrak{t} + \tilde{\eta}\mathfrak{p} = 0$ and thus

$$(\tilde{\eta}\mathfrak{p}, \mathfrak{q}) = (d\mathfrak{t}, \mathfrak{q}) = 0.$$

Hence,

$$\tilde{\eta} = \mathfrak{f} \wedge d\mathfrak{f} \circ A - \mathfrak{t} \wedge d\mathfrak{t}, \tag{6.4}$$

for some $A \in \Gamma End(T\Sigma)$. By Lemma 3.34, T_1 and T_2 are eigenspaces of A. Let α_1 and α_2 denote the corresponding eigenvalues. Now the quadratic differential of $\tilde{\eta}$ is given by

$$q = -(d\mathfrak{f} \circ A, d\mathfrak{f}) + (d\mathfrak{t}, d\mathfrak{t}).$$

Recall from Section 4.2 that q is a divergence-free quadratic differential. Thus, in terms of arbitrary curvature line coordinates (u, v), $q = -\epsilon^2 U^2 du^2 + V^2 dv^2$, for some smooth functions U of u and V of v. Therefore,

$$-\epsilon^2 U^2 = -\alpha_1 E + \kappa_1^2 E \quad \text{and} \quad V^2 = -\alpha_2 G + \kappa_2^2 G. \tag{6.5}$$

Furthermore, since $\tilde{\eta}$ is closed, we have that

$$0 = d\mathfrak{f} \curlywedge d\mathfrak{f} \circ A - d\mathfrak{t} \curlywedge d\mathfrak{t} = (\alpha_1 + \alpha_2 - 2K)(\mathfrak{f}_u \wedge \mathfrak{f}_v) du \wedge dv. \tag{6.6}$$

Substituting α_1 and α_2 from (6.5) into (6.6) yields

$$EG(\kappa_1 - \kappa_2)^2 = V^2 E - \epsilon^2 U^2 G.$$

Now by choosing curvature line coordinates so that $q = -\epsilon^2 du^2 + dv^2$ we have that

$$EG(\kappa_1 - \kappa_2)^2 = E - \epsilon^2 G. \tag{6.7}$$

Comparing this with [20], we see that in the case that the isothermic sphere congruences of f are real, i.e., q is indefinite, $\mathfrak{f}$ is a Guichard surface of the first kind in the space form $\mathfrak{Q}^3$, whereas in the case that they are complex, $\mathfrak{f}$ is a Guichard surface of the second kind.

Conversely, if (6.7) is satisfied in $\mathfrak{Q}^3$ for some curvature line coordinates (u, v), then let

$$\tilde{\eta} := \mathfrak{f} \wedge ((\tfrac{\epsilon^2}{E} + \kappa_1^2)\mathfrak{f}_u du + (-\tfrac{1}{G} + \kappa_2^2)\mathfrak{f}_v dv) - \mathfrak{t} \wedge d\mathfrak{t}.$$

One can check that $\tilde{\eta}$ is closed and it is clear that $\tilde{p}(t) = \mathfrak{p} - t\mathfrak{t}$ is a linear conserved quantity for $d + t\tilde{\eta}$.

We thus have the following theorem:

Theorem 6.24. *Special Ω-surfaces of type one whose linear conserved quantity p satisfies $(p(t), p(t))$ being a linear polynomial with non-zero constant term are the surfaces that project to Guichard surfaces in any space form with point sphere complex $p(0)$.*

Remark 6.25. *Guichard surfaces were given a conformally invariant treatment by Burstall and Calderbank in [6, 21] as Möbius-flat hypersurfaces. This treatment has been studied further in [27, 28, 29].*

The associate surface

Suppose now that $|\mathfrak{p}|^2 = -1$ and choose a null space form vector $\mathfrak{q}_\infty \in \langle \mathfrak{p} \rangle^\perp$. Then $\mathfrak{Q}^3$ is isometric to Euclidean 3-space. Let $\nu : \Sigma \to \mathbb{R}^3$ denote a corresponding projection of f with Gauss map $n : \Sigma \to S^2$.

Guichard surfaces were originally defined [49] in Euclidean space as the surfaces $\nu : \Sigma \to \mathbb{R}^3$ for which there exists another surface $\nu^g : \Sigma \to \mathbb{R}^3$ with the same spherical representation as ν such that the principal curvatures of these surfaces satisfy

$$\frac{1}{\kappa_1 \kappa_2^g} + \frac{1}{\kappa_2 \kappa_1^g} = -2.$$

Now

$$\tilde{\eta} := \mathfrak{f} \wedge ((\tfrac{\epsilon^2}{E} + \kappa_1^2)\mathfrak{f}_u du + (-\tfrac{1}{G} + \kappa_2^2)\mathfrak{f}_v dv) - \mathfrak{t} \wedge d\mathfrak{t}$$

and $(\tilde{\eta}\mathfrak{p}, \mathfrak{q}_\infty) = 0$. Therefore, by comparing with Section 4.3, there exists an associate surface ν^D of ν such that the associate Gauss map is given by $\hat{\nu} = n$. Thus

$$0 = \frac{1}{\kappa_1 \kappa_2^D} + \frac{1}{\kappa_2 \kappa_1^D} - \frac{1}{\hat{\kappa}_1} - \frac{1}{\hat{\kappa}_2} = \frac{1}{\kappa_1 \kappa_2^D} + \frac{1}{\kappa_2 \kappa_1^D} + 2.$$

Hence, ν^D is an associate surface in the sense of Guichard [49].

Calapso transforms

Let $f^t := T(t)f$ be a Calapso transform of f. Then by Proposition 6.14, the middle connection of f^t admits a linear conserved quantity p^t defined by

$$p^t(s) = T(t)p(t+s).$$

Now since $T(t)$ take values in $\mathrm{O}(4,2)$, we have that

$$(p^t(s), p^t(s)) = (p(t+s), p(t+s)).$$

Therefore, $(p^t(s), p^t(s))$ is a linear polynomial with constant term $(p(t), p(t))$. Since $(p(t), p(t))$ is a linear polynomial in t with non-zero constant term, it admits a single root which we shall denote t_0. By applying Theorem 6.24, we obtain the following theorem:

Theorem 6.26. *If $t \neq t_0$, then the Calapso transform f^t projects to Guichard surfaces in any space form with point sphere complex $T(t)p(t)$.*

Remark 6.27. *The Calapso transform f^{t_0} admits a linear polynomial p^{t_0} such that $(p^{t_0}(s), p^{t_0}(s))$ is a linear polynomial with vanishing constant term. Therefore Theorem 6.24 does not apply in this case.*

The Eisenhart transformation

In [37], Eisenhart develops a transformation for Guichard surfaces in Euclidean 3-space akin to the Darboux transformation for isothermic surfaces. We shall show that this transformation is induced by the Darboux transformation of Chapter 5. Firstly let us recall the construction of [37]:

Suppose that $\nu : \Sigma \to \mathbb{R}^3$ is a Guichard surface with unit normal N and let (u, v) be the special curvature line coordinates such that[3]

$$EG(\kappa_1 - \kappa_2)^2 = E - \epsilon^2 G.$$

Let $e_1 := \frac{\nu_u}{||\nu_u||}$ and $e_2 := \frac{\nu_v}{||\nu_v||}$. Let θ and ξ be functions such that

$$\sqrt{E} = e^{\xi} \cosh \epsilon\theta \quad \text{and} \quad \sqrt{G} = e^{\xi} \sinh \epsilon\theta.$$

In [37], it is shown that there exists a function h such that

$$\kappa_1 = e^{-\xi}(\epsilon^2 \tanh \epsilon\theta + h) \quad \text{and} \quad \kappa_2 = e^{-\xi}(\coth \epsilon\theta + h).$$

Then an Eisenhart transformation of ν with parameter m is

$$\hat{\nu} := \nu - \frac{1}{\sigma m}(\mu N + \alpha e_1 + \beta e_2)$$

[3]Note that the roles of u and v here are opposite to that of [37].

with unit normal

$$\hat{N} := N + \frac{\mu}{\lambda}(\hat{\nu} - \nu),$$

where α, β, σ, μ and λ are functions satisfying the fundamental quadratic equation

$$2\lambda\sigma m = \alpha^2 + \beta^2 + \mu^2 \tag{6.8}$$

and the fundamental system of equations

$$\begin{aligned}
d\sigma &= \alpha\, e^{-\xi}(t\psi + \phi)\, du + \beta\, e^{-\xi}(t\phi + \epsilon^2\psi)\, dv \\
d\lambda &= \alpha\, e^{\xi}\cosh\epsilon\theta\, du + \beta\, e^{\xi}\sinh\epsilon\theta\, dv \\
d\mu &= -\alpha\,(\epsilon^2\sinh\epsilon\theta + h\cosh\epsilon\theta)\, du - \beta\,(\cosh\epsilon\theta + h\sinh\epsilon\theta)\, dv \\
d\alpha &= \beta\,((-\coth\epsilon\theta\tfrac{\partial\xi}{\partial v} + \epsilon^2\tfrac{\partial\theta}{\partial v})du + (\tanh\epsilon\theta\tfrac{\partial\xi}{\partial u} + \tfrac{\partial\theta}{\partial u})dv) + \mu\,(\epsilon^2\sinh\epsilon\theta + h\cosh\epsilon\theta)du \\
&+ \sigma\, me^{\xi}\cosh\epsilon\theta du + \lambda\, me^{-\xi}(t\psi + \phi)du, \\
d\beta &= \alpha\,((\coth\epsilon\theta\tfrac{\partial\xi}{\partial v} + \epsilon^2\tfrac{\partial\theta}{\partial v})du - (\tanh\epsilon\theta\tfrac{\partial\xi}{\partial u} + \tfrac{\partial\theta}{\partial u})dv) + \mu\,(\cosh\epsilon\theta + h\sinh\epsilon\theta)dv \\
&+ \sigma\, me^{\xi}\sinh\epsilon\theta dv + \lambda\, me^{-\xi}(t\phi + \psi)dv
\end{aligned}$$

where

$$\phi = \cosh\epsilon\theta + (h + \tfrac{\mu}{\lambda}e^{\xi})\sinh\theta, \quad \psi = \epsilon^2\sinh\epsilon\theta + (h + \tfrac{\mu}{\lambda}e^{\xi})\cosh\theta, \quad \text{and} \quad t = h + \tfrac{\mu}{\lambda}e^{\xi}.$$

This system is non-linear, however Calderbank provided a way to linearise this system in [9] by replacing σ with $\gamma := \sigma + \frac{\mu^2}{\lambda}$. We then have a linear system

$$d\begin{pmatrix}\alpha\\ \beta\\ \lambda\\ \mu\\ \gamma\end{pmatrix} = A\begin{pmatrix}\alpha\\ \beta\\ \lambda\\ \mu\\ \gamma\end{pmatrix}, \tag{6.9}$$

where

$$A := \begin{pmatrix}
0 & -\zeta & m(\sqrt{E}\kappa_1^2 + \frac{\epsilon^2}{\sqrt{E}})du & (2m+1)\sqrt{E}\kappa_1 du & m\sqrt{E}du \\
\zeta & 0 & m(\sqrt{G}\kappa_2^2 - \frac{1}{\sqrt{G}})dv & (2m+1)\sqrt{G}\kappa_2 dv & m\sqrt{G}dv \\
\sqrt{E}du & \sqrt{G}dv & 0 & 0 & 0 \\
-\sqrt{E}\kappa_1 du & -\sqrt{G}\kappa_2 dv & 0 & 0 & 0 \\
(\sqrt{E}\kappa_1^2 + \frac{\epsilon^2}{\sqrt{E}})du & (\sqrt{G}\kappa_2^2 - \frac{1}{\sqrt{G}})dv & 0 & 0 & 0
\end{pmatrix}$$

with

$$\zeta := \sqrt{\tfrac{E}{G}}(ln\sqrt{E})_v du - \sqrt{\tfrac{G}{E}}(ln\sqrt{G})_u dv.$$

We shall now construct a frame of $\underline{\mathbb{R}}^{4,2}$: let $\mathfrak{q}_\infty$ and $\mathfrak{p}$ define a space form vector and point sphere complex for a Euclidean space form $\mathfrak{Q}^3$. Let $\mathfrak{q}_0 \in \langle\mathfrak{p}\rangle^\perp$ be a complementary null vector for $\mathfrak{q}_\infty$. Then we may identify ν with $\mathfrak{f}:\Sigma\to\mathfrak{Q}^3$ via

$$\mathfrak{f}=\nu+\mathfrak{q}_0+\frac{1}{2}(\nu,\nu)\mathfrak{q}_\infty.$$

Let $\tilde{e}_1:=\frac{\mathfrak{f}_u}{||\mathfrak{f}_u||}$ and $\tilde{e}_2:=\frac{\mathfrak{f}_v}{||\mathfrak{f}_v||}$, and let $\mathfrak{t}:\Sigma\to\mathfrak{P}^3$ be the tangent plane congruence of $\mathfrak{f}$. Then

$$\tilde{e}_1=e_1+(e_1,\nu)\mathfrak{q}_\infty,\quad \tilde{e}_2=e_2+(e_2,\nu)\mathfrak{q}_\infty \quad\text{and}\quad \mathfrak{t}=N+(N,\sigma)\mathfrak{q}_\infty+\mathfrak{p}.$$

Let $\mathcal{F}:=(\mathfrak{f},\mathfrak{t},\tilde{e}_1,\tilde{e}_2,\mathfrak{q}_\infty,\mathfrak{p})$ be a frame for $\underline{\mathbb{R}}^{4,2}$. Then $d\mathcal{F}=\mathcal{F}\omega$, where

$$\omega=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & \sqrt{E}\kappa_1du & \sqrt{G}\kappa_2dv & 0 & 0\\ \sqrt{E}du & -\sqrt{E}\kappa_1du & 0 & \zeta & 0 & 0\\ \sqrt{G}dv & -\sqrt{G}\kappa_2dv & -\zeta & 0 & 0 & 0\\ 0 & 0 & \sqrt{E}du & \sqrt{G}dv & 0 & 0\\ 0 & 0 & -\sqrt{E}\kappa_1du & -\sqrt{G}\kappa_2dv & 0 & 0 \end{pmatrix}.$$

Now the lift $\hat{\mathfrak{f}}:\Sigma\to\mathfrak{Q}^3$ of $\hat{\nu}$ is given by

$$\begin{aligned}\hat{\mathfrak{f}} &= \hat{\nu}+\mathfrak{q}_0+\frac{1}{2}(\hat{\nu},\hat{\nu})\mathfrak{q}_\infty\\ &= \nu-\frac{1}{\sigma m}(\mu N+\alpha e_1+\beta e_2)+\mathfrak{q}_0\\ &+ \frac{1}{2}((\nu,\nu)+\frac{2}{\sigma m}(\nu,\mu N+\alpha e_1+\beta e_2)+\frac{1}{\sigma^2m^2}(\mu^2+\alpha^2+\beta^2))\mathfrak{q}_\infty\\ &= \mathfrak{f}-\frac{1}{\sigma m}(\mu\mathfrak{t}+\alpha\tilde{e}_1+\beta\tilde{e}_2-\lambda\mathfrak{q}_\infty-\mu\mathfrak{p})\end{aligned}$$

and its tangent plane congruence is given by

$$\hat{\mathfrak{t}}=\mathfrak{t}+\frac{\mu}{\lambda}(\hat{\mathfrak{f}}-\mathfrak{f}).$$

Let

$$\begin{aligned}\hat{\sigma} &:= \gamma\hat{\mathfrak{f}}-\mu\hat{\mathfrak{t}}\\ &= \gamma\mathfrak{f}-\frac{1}{m}(\mu(1+m)\mathfrak{t}+\alpha\tilde{e}_1+\beta\tilde{e}_2-\lambda\mathfrak{q}_\infty-\mu\mathfrak{p})\\ &= \mathcal{F}w\end{aligned}$$

where $w:=(\gamma,-\frac{\mu(1+m)}{m},-\frac{\alpha}{m},-\frac{\beta}{m},\frac{\lambda}{m},\frac{\mu}{m})^t$. On the other hand if we let $\hat{\sigma}\in\Gamma\underline{\mathbb{R}}^{4,2}$, and assume that $(\hat{\sigma},p(-m))=0$, then $\hat{\sigma}$ has the form

$$\hat{\sigma}=\gamma\mathfrak{f}-\frac{1}{m}(\mu(1+m)\mathfrak{t}+\alpha\tilde{e}_1+\beta\tilde{e}_2-\lambda\mathfrak{q}_\infty-\mu\mathfrak{p}),$$

for some smooth functions α, β, γ, λ and μ. We then have that $\hat{\sigma}$ is lightlike if and only if the fundamental quadratic equation (6.8) is satisfied. Now in terms of the frame $\mathcal{F}$ we have that

$$\begin{aligned} \eta &= \mathfrak{f} \wedge ((\sqrt{E}\kappa_1^2 + \tfrac{\epsilon^2}{\sqrt{E}})\tilde{e}_1 du + (\sqrt{G}\kappa_2^2 - \tfrac{1}{\sqrt{G}})\tilde{e}_2 dv) - \mathfrak{t} \wedge d\mathfrak{t} \\ &= \mathcal{F} B \mathcal{F}^{-1}, \end{aligned}$$

where

$$B = \begin{pmatrix} 0 & 0 & -(\sqrt{E}\kappa_1^2+\frac{\epsilon^2}{\sqrt{E}})du & -(\sqrt{G}\kappa_2^2-\frac{1}{\sqrt{G}})dv & 0 & 0 \\ 0 & 0 & -\sqrt{E}\kappa_1 du & -\sqrt{G}\kappa_2 dv & 0 & 0 \\ 0 & 0 & 0 & 0 & -(\sqrt{E}\kappa_1^2+\frac{\epsilon^2}{\sqrt{E}})du & -\sqrt{E}\kappa_1 du \\ 0 & 0 & 0 & 0 & -(\sqrt{G}\kappa_2^2-\frac{1}{\sqrt{G}})dv & -\sqrt{G}\kappa_2 dv \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then $\hat{\sigma}$ is a parallel section of $d - m\eta$ if and only if

$$0 = (d - m\eta)\hat{\sigma} = \mathcal{F}(dw + (\omega - mB)w).$$

That is w is a parallel section of $d + \omega - mB$. Now

$$\omega - mB = \begin{pmatrix} 0 & 0 & m(\sqrt{E}\kappa_1^2+\frac{\epsilon^2}{\sqrt{E}})du & m(\sqrt{G}\kappa_2^2-\frac{1}{\sqrt{G}})dv & 0 & 0 \\ 0 & 0 & (1+m)\sqrt{E}\kappa_1 du & (1+m)\sqrt{G}\kappa_2 dv & 0 & 0 \\ \sqrt{E}du & -\sqrt{E}\kappa_1 du & 0 & \zeta & m(\sqrt{E}\kappa_1^2+\frac{\epsilon^2}{\sqrt{E}})du & m\sqrt{E}\kappa_1 du \\ \sqrt{G}dv & -\sqrt{G}\kappa_2 dv & -\zeta & 0 & m(\sqrt{G}\kappa_2^2-\frac{1}{\sqrt{G}})dv & m\sqrt{G}\kappa_2 dv \\ 0 & 0 & \sqrt{E}du & \sqrt{G}dv & 0 & 0 \\ 0 & 0 & -\sqrt{E}\kappa_1 du & -\sqrt{G}\kappa_2 dv & 0 & 0 \end{pmatrix}$$

and one can then check that $w = (\gamma, -\frac{\mu(1+m)}{m}, -\frac{\alpha}{m}, -\frac{\beta}{m}, \frac{\lambda}{m}, \frac{\mu}{m})^t$ being a parallel section of $d + (\omega - mB)$ is equivalent to the fundamental system of equations (6.9) being satisfied. We have therefore proved the following theorem:

Theorem 6.28. *The Darboux transforms $\hat{f}$ of f with parameter $-m$ such that $p(-m) \in \Gamma\hat{s}^{\perp}$ are the Eisenhart transformations with parameter m in Euclidean geometries with point sphere complex $\mathfrak{p} = p(0)$.*

6.3.3 L-isothermic surfaces

L-isothermic surfaces were originally discovered by Blaschke [3] and have been the subject of interest recently in for example [53, 54, 55, 57, 67]. They are the surfaces in $\mathbb{R}^3$ that admit curvature line coordinates that are conformal with respect to the third fundamental form of the surface, or as Musso and Nicolodi [55] put it, there exists a holomorphic[4] (with respect to the third fundamental form) quadratic differential that commutes with the second fundamental form, i.e., if we use the complex structure induced on Σ by $I\!I\!I$ to split the second fundamental form into bidegrees,

$$I\!I = I\!I^{2,0} + I\!I^{1,1} + I\!I^{0,2},$$

[4]That is, locally there exists a complex coordinate z on Σ such that $q^{2,0} := q(\frac{\partial}{\partial z}, \frac{\partial}{\partial z})dz^2 = dz^2$ and $I\!I\!I = e^{2u} dz d\bar{z}$.

then $I\!I^{2,0} = \mu q^{2,0}$, for some real valued function $\mu : \Sigma \to \mathbb{R}$. In [55], L-isothermic surfaces were also characterised in terms of the standard model for Laguerre geometry $\mathbb{R}^{3,1}$ (see for example [3, 25]). In this subsection we will show that Legendre lifts of L-isothermic surfaces are the special Ω-surfaces of type one whose linear conserved quantity p satisfies $(p(t), p(t)) = 0$.

Recall from Subsection 2.4.2 that a non-zero lightlike vector $\mathfrak{q}_\infty$ defines a Laguerre geometry and that given $\mathfrak{q}_0 \in \mathcal{L}$ such that $(\mathfrak{q}_0, \mathfrak{q}_\infty) = -1$ and $\mathfrak{p} \in \langle \mathfrak{q}_0, \mathfrak{q}_\infty \rangle^\perp$ such that $|\mathfrak{p}|^2 = -1$, then $\mathfrak{Q}^3$ is isometric to $\mathbb{R}^3$.

Now suppose that $f : \Sigma \to \mathcal{Z}$ is a Legendre map and that f projects to a surface $\mathfrak{f} : \Sigma \to \mathfrak{Q}^3$ with tangent plane congruence $\mathfrak{t} : \Sigma \to \mathfrak{P}^3$. Then

$$\mathfrak{f} = \nu + \mathfrak{q}_0 + \frac{1}{2}(\nu, \nu)\mathfrak{q}_\infty \quad \text{and} \quad \mathfrak{t} = n + \mathfrak{p} + (n, \nu)\mathfrak{q}_\infty,$$

where $\nu : \Sigma \to \mathbb{R}^3$ is the corresponding surface in $\mathbb{R}^3$ with unit normal $n : \Sigma \to S^2$. Suppose that there exists a holomorphic (with respect to the third fundamental form of ν, $I\!I\!I = (dn, dn)$) quadratic differential that commutes with the second fundamental form of ν, $I\!I = -(d\nu, dn)$. This implies that if we let $Q \in \Gamma End(T\Sigma)$ such that

$$q = (dn, dn \circ Q)$$

then Q is trace-free and symmetric with respect to $I\!I\!I$ and the two-tensor

$$(d\nu, dn \circ Q)$$

is symmetric. Now let

$$\begin{aligned} \eta &:= \mathfrak{t} \wedge d\mathfrak{t} \circ Q \\ &= (n + \mathfrak{p} + (n, \nu)\mathfrak{q}_\infty) \wedge (dn \circ Q + (dn \circ Q, \nu)\mathfrak{q}_\infty). \end{aligned}$$

Then

$$d\eta = dn \curlywedge dn \circ Q + \mathfrak{q}_\infty \wedge ((dn \curlywedge dn \circ Q)\nu) + (n + \mathfrak{p} + (n, \nu)\mathfrak{q}_\infty) \wedge (d(dn \circ Q) + d(dn \circ Q, \nu)\mathfrak{q}_\infty).$$

It follows from the fact that Q is trace-free that $dn \curlywedge dn \circ Q = 0$. Furthermore, one can check that q being holomorphic implies that $dn \circ Q$ is closed. Finally, for any $X, Y \in \Gamma T\Sigma$,

$$d(dn \circ Q, \nu)(X, Y) = (dn \circ Q(X), d_Y \nu) - (dn \circ Q(Y), d_X \nu) = 0,$$

since $(d\nu, dn \circ Q)$ is symmetric. Therefore, η is closed. Moreover,

$$q(X, Y) = (dn, dn \circ Q) = tr(\sigma \to \eta(X) d_Y \sigma)$$

is non-degenerate and $\eta \mathfrak{q}_\infty = 0$. Hence, f is an Ω-surface and for some $\tau \in \Gamma \wedge^2 f$, $p(t) :=$

$\exp(t\tau)\mathfrak{q}_\infty$ is a linear conserved quantity of the middle connection of f satisfying

$$(p(t), p(t)) = (\mathfrak{q}_\infty, \mathfrak{q}_\infty) = 0.$$

Conversely, suppose that f is a special Ω-surface of type one whose linear conserved quantity p satisfies $(p(t), p(t)) = 0$. Let $\mathfrak{q}_\infty := p_0$. Then $\mathfrak{q}_\infty$ is a space form vector for a space form with vanishing sectional curvature. Furthermore, by Corollaries 6.8 and 6.9, one of the isothermic sphere congruences, without loss of generality s^+, takes values in $\langle \mathfrak{q}_\infty \rangle^\perp$. Let $\mathfrak{p} \in \langle \mathfrak{q}_\infty \rangle^\perp$ be a point sphere complex with $|\mathfrak{p}|^2 = -1$ and let $\mathfrak{t} \in \Gamma s^+$ be the lift of s^+ such that $(\mathfrak{t}, \mathfrak{p}) = -1$. Then $\mathfrak{t}$ defines a tangent plane congruence for the space form projection $\mathfrak{f} : \Sigma \to \mathfrak{Q}^3$ of f. Now η^+ has the form

$$\eta^+ = \mathfrak{t} \wedge d\mathfrak{t} \circ Q,$$

for some $Q \in \Gamma End(T\Sigma)$. Therefore,

$$q(X, Y) = tr(\sigma \mapsto \eta^+_X d_Y \sigma) = (d\mathfrak{t}, d\mathfrak{t} \circ Q),$$

and q is holomorphic with respect to the conformal structure induced by $\mathfrak{t}$. Furthermore, since η^+ is closed, we have that

$$\begin{aligned} 0 = (d\eta^+(X,Y))\mathfrak{f} &= (d\mathfrak{t} \curlywedge d\mathfrak{t} \circ Q + \mathfrak{t} \wedge d(d\mathfrak{t} \circ Q))(X,Y)\mathfrak{f} \\ &= -(d(d\mathfrak{t} \circ Q)(X,Y), \mathfrak{f})\mathfrak{t} \\ &= ((d\mathfrak{t} \circ Q(X), d_Y \mathfrak{f}) - (d\mathfrak{t} \circ Q(Y), d_X \mathfrak{f}))\mathfrak{t}. \end{aligned}$$

Thus, q commutes with the second fundamental form of $\mathfrak{f}$. Hence, $\mathfrak{f}$ projects to an L-isothermic surface.

We therefore have the following theorem:

Theorem 6.29. *Special Ω-surfaces of type one whose linear polynomial conserved quantity p satisfies $(p(t), p(t)) = 0$ are the L-isothermic surfaces of any Laguerre geometry defined by $p(0)$.*

Calapso transforms

L-isothermic surfaces are well known to be the deformable surfaces of Laguerre geometry [53] and this gives rise to T-transforms for these surfaces [57]. Therefore it is unsurprising that the Calapso transforms of Legendre lifts of L-isothermic surfaces yield L-isothermic surfaces.

Fix $t \in \mathbb{R}$ and let f^t be a Calapso transform of f. Then by Proposition 6.14 the middle connection of f^t admits a linear conserved quantity p^t defined by

$$p^t(s) = T(t)p(t+s).$$

Since $T(t)$ takes values in $\mathrm{O}(4,2)$ we have that

$$(p^t(s), p^t(s)) = (p(t+s), p(t+s)) = 0.$$

Now $T(t)p(t)$ is a constant null vector. Therefore, by premultiplying by an appropriate Lie sphere transformation, we may assume that it is $\mathfrak{q}_\infty$. By applying Theorem 6.29 we obtain the following theorem:

Theorem 6.30. *The Calapso transforms of L-isothermic surfaces are L-isothermic.*

The Bianchi-Darboux transform

In [55] a transformation was developed for L-isothermic surfaces called the Bianchi-Darboux transformation, which is analogous to the Darboux transformation for isothermic surfaces. We shall show how this corresponds to the Darboux transformation of Chapter 5.

Firstly we will use the frame $\mathcal{F} = (\frac{\mathfrak{p}+n}{\sqrt{2}}, e_1, e_2, \frac{\mathfrak{p}-n}{\sqrt{2}}, q_\infty, q_0)$ for $\underline{\mathbb{R}}^{4,2}$, where $e_1 = \frac{\nu_u}{||\nu_u||}$ and $e_2 = \frac{\nu_v}{||\nu_v||}$ and (u,v) are curvature line coordinates for ν that are conformal with respect to the third fundamental form. Let $e^{2\phi}$ be the conformal factor of the third fundamental form. Then $d\mathcal{F} = \mathcal{F}\omega$, where

$$\omega = \begin{pmatrix} 0 & -\frac{e^\phi}{\sqrt{2}}du & -\frac{e^\phi}{\sqrt{2}}dv & 0 & 0 & 0 \\ \frac{e^\phi}{\sqrt{2}}du & 0 & \phi_v du - \phi_u dv & -\frac{e^\phi}{\sqrt{2}}du & 0 & 0 \\ \frac{e^\phi}{\sqrt{2}}dv & -\phi_v du + \phi_u dv & 0 & -\frac{e^\phi}{\sqrt{2}}dv & 0 & 0 \\ 0 & \frac{e^\phi}{\sqrt{2}}du & \frac{e^\phi}{\sqrt{2}}dv & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The tangent plane congruence of ν is given by $\mathfrak{t} = n + \mathfrak{p} + (n,\nu)\mathfrak{q}_\infty$ and

$$\eta^+ = \mathfrak{t} \wedge e^{-2\phi}(\mathfrak{t}_u du - \mathfrak{t}_v dv).$$

Therefore in terms of our framing $\eta^+ = \mathcal{F}B\mathcal{F}^{-1}$, where

$$B := \begin{pmatrix} 0 & -\sqrt{2}e^{-\phi}du & \sqrt{2}e^{-\phi}dv & 0 & 0 & \sqrt{2}e^{-\phi}(e_1 du - e_2 dv, \nu) \\ 0 & 0 & 0 & -\sqrt{2}e^{-\phi}du & 0 & -e^{-\phi}(n,\nu)du \\ 0 & 0 & 0 & \sqrt{2}e^{-\phi}dv & 0 & e^{-\phi}(n,\nu)dv \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -e^{-\phi}(n,\nu)du & e^{-\phi}(n,\nu)dv & \sqrt{2}e^{-\phi}(-e_1 du + e_2 dv, \nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Now suppose that $\hat{\sigma}^+ : \Sigma \to \mathcal{L}$ takes values in $\langle q_\infty \rangle^\perp$. Then $\hat{\sigma}^+ = \mathcal{F}w$ where $w = (w^1, ..., w^5, 0)^t$. Then $\hat{\sigma}^+$ is a parallel section of $d - \frac{m}{2}\eta^+$ for some $m \in \mathbb{R}^\times$ if and only if v is a parallel section of $d+\omega-\frac{m}{2}B$, i.e.,

$$\begin{aligned}
dw^1 &= \tfrac{e^\phi-me^{-\phi}}{\sqrt{2}}du\,w^2+\tfrac{e^\phi+me^{-\phi}}{\sqrt{2}}dv\,w^3\\
dw^2 &= -\tfrac{e^\phi}{\sqrt{2}}du\,w^1-(\phi_v du-\phi_u dv)\,w^3+\tfrac{e^\phi-me^{-\phi}}{\sqrt{2}}du\,w^4\\
dw^3 &= -\tfrac{e^\phi}{\sqrt{2}}dv\,w^1+(\phi_v du-\phi_u dv)\,w^2+\tfrac{e^\phi+me^{-\phi}}{\sqrt{2}}dv\,w^4\\
dw^4 &= -\tfrac{e^\phi}{\sqrt{2}}du\,w^2-\tfrac{e^\phi}{\sqrt{2}}dv\,w^3
\end{aligned}$$

and

$$dw^5 = \tfrac{me^{-\phi}}{\sqrt{2}}(-\tfrac{1}{\sqrt{2}}(n,\nu)du\,w^2+\tfrac{1}{\sqrt{2}}(n,\nu)dv\,w^3+(-e_1du+e_2dv,\nu)\,w^4) \tag{6.10}$$

A lengthy computation shows that if we let

$$r:=(e_1,\nu)\,w^2+(e_2,\nu)\,w^3+\tfrac{1}{\sqrt{2}}(n,\nu)\,(w^1-w^4)-w^5$$

then we may replace (6.10) with

$$dr=(e_1,\nu_u)du\,w^2+(e_2,\nu_v)dv\,w^3.$$

This system of equations is then equivalent to the system given in [55, Section 5.1]. Now assume that $\hat{\sigma}^+$ never belongs to f and let $s_0:=f\cap\langle\hat{\sigma}^+\rangle^\perp$ and $\hat{f}:=s_0\oplus\langle\hat{\sigma}^+\rangle$. If we assume that $w^1+w^4>0$ (this equates to the assumption in [55] that $(w^1,w^2,w^3,w^4)^t$ takes values in the positive lightcone of $\mathbb{R}^{3,1}$) then $(\hat{\sigma}^+,\mathfrak{t})$ is nowhere zero and thus $\mathfrak{t}$ never belongs to s_0. Therefore, let

$$\sigma_0:=\mathfrak{f}+\mu\mathfrak{t}\in\Gamma s_0.$$

Then $\sigma_0\in\Gamma\langle\hat{\sigma}^+\rangle^\perp$ implies that

$$\mu=\frac{1}{\sqrt{2}w^4}\left(w^2(e_1,\nu)+w^3(e_2,\nu)+\frac{1}{\sqrt{2}}(w^1-w^4)(n,\nu)-w^5\right)=\frac{r}{\sqrt{2}w^4}.$$

One can then check that the point sphere map $\hat{\mathfrak{f}}:=\hat{f}\cap\mathfrak{Q}^3$ is given by

$$\hat{\mathfrak{f}}=-\frac{r}{w^4(w^1+w^4)}\hat{\sigma}^++\sigma_0,$$

and thus the Euclidean projection of $\hat{f}$ in $\langle\mathfrak{q}_0,\mathfrak{q}_\infty,\mathfrak{p}\rangle^\perp$ is given by

$$\begin{aligned}
\hat{\nu} &= \hat{\mathfrak{f}}+\mathfrak{q}_0+(\hat{\mathfrak{f}},\mathfrak{q}_0)\mathfrak{q}_\infty\\
&= \nu+\frac{r}{w^4(w^1+w^4)}(w^4n-w^2e_1-w^3e_2).
\end{aligned}$$

Comparing this with [55, Theorem 4] yield the following result:

Theorem 6.31. *Darboux transforms $\hat{f}$ of f with parameter $-m/2$ such that $p(-m/2)\in\Gamma\hat{s}^\perp$*

are the Legendre lifts of the Bianchi-Darboux transforms with parameter m in the Laguerre geometry defined by $p(0)$.

Associate surface

We shall now recover the result of [66, Section 6] that L-isothermic surfaces are the Combescure transforms of minimal surfaces.

Let $\nu : \Sigma \to \mathbb{R}^3$ be an L-isothermic surface. Given that

$$\eta^+ = \mathfrak{t} \wedge d\mathfrak{t} \circ Q$$

for some $Q \in \Gamma End(T\Sigma)$, we have that $(\eta^+ \mathfrak{q}_\infty, \mathfrak{p}) = 0$. By comparing with Section 4.3, we have that there is an associate Gauss map $\hat{\nu}$ of ν satisfying

$$0 = \frac{1}{\hat{\kappa}_1} + \frac{1}{\hat{\kappa}_2} = \frac{\hat{\kappa}_1 + \hat{\kappa}_2}{\hat{\kappa}_1 \hat{\kappa}_2}.$$

Thus, there exists a minimal surface $\hat{\nu}$ with the same spherical representation as ν. In fact, we have a converse to this result:

Theorem 6.32. *Suppose that $\hat{\nu} : \Sigma \to \mathbb{R}^3$ is a minimal surface. Then any Combescure transform $\nu : \Sigma \to \mathbb{R}^3$ of $\hat{\nu}$ is an L-isothermic surface.*

Proof. Let ν be a Combescure transformation of $\hat{\nu}$, i.e., ν and $\hat{\nu}$ have the same spherical representation. Let n be the common normal of these surfaces. Then the result follows by the fact that

$$\eta := (n + \mathfrak{p} + (n, \nu)\mathfrak{q}_\infty) \wedge (d\hat{\nu} + (d\hat{\nu}, \nu)\mathfrak{q}_\infty)$$

is a closed one-form. □

Remark 6.33. *The characterisation of L-isothermic surfaces as the Combescure transforms of minimal surfaces shows that the class of L-isothermic surfaces is preserved by Combescure transformation.*

Further work

There is one case that we have not considered in this section - when f admits a linear conserved quantity p such that $(p(t), p(t))$ is a linear polynomial with vanishing constant term. It would be interesting to know if these surfaces have a classical interpretation in the Laguerre geometry defined by $p(0)$. One interesting fact about these surfaces is that if we further project into a Euclidean subgeometry of $\langle p(0) \rangle^\perp$ then the resulting surface is an associate surface of itself. Furthermore, by Remark 6.27 these surfaces appear as one of the Calapso transforms of a Guichard surface.

6.3.4 Complementary surfaces

Suppose that f is a special Ω-surface of type one with linear conserved quantity $p(t) = p_0 + tp_1$. Now the polynomial $(p(t), p(t))$ has degree less than or equal to one and admits non-zero roots if and only if either

- $(p(t), p(t))$ is linear with non-zero constant term, in which case f projects to a Guichard surface in $\langle p(0)\rangle^\perp$, by Theorem 6.24, or
- $(p(t), p(t))$ is the zero polynomial, in which case f projects to an L-isothermic surface in the Laguerre geometry defined by $p(0)$, by Theorem 6.29.

Now suppose that m is a root of $p(m)$ and let $\hat{f}$ be the corresponding complementary surface. Now by Theorem 6.4, $p_1 \in \Gamma f$ and thus

$$f + \hat{f} = f \oplus \langle p(0)\rangle.$$

Conversely, suppose that $\hat{f}$ is a Darboux transform of f with parameter m such that there exists a constant vector $\mathfrak{q} \in \Gamma(f + \hat{f})$. Let $\hat{\sigma} \in \Gamma\hat{f}$ be a parallel section of $d + m\eta^m$. Now

$$\hat{\sigma} = \lambda\,\mathfrak{q} + \sigma$$

for some non-zero smooth function λ and $\sigma \in \Gamma f$. Thus,

$$0 = (d + m\eta^m)\hat{\sigma} = d\lambda\,\mathfrak{q} + d\sigma + m\lambda\eta^m\mathfrak{q}.$$

Since $\mathfrak{q}$ never belongs to f and $\mathfrak{q}$ belongs to $f + \hat{f}$, we have that $\mathfrak{q}$ never belongs to $f^\perp$. Thus,

$$d\lambda = 0 \quad \text{and} \quad d\sigma + m\lambda\eta^m\mathfrak{q} = 0.$$

Therefore, $d + t\eta^m$ admits a linear conserved quantity p defined by

$$p(t) = m\lambda\mathfrak{q} + t\sigma.$$

Furthermore, using that $\hat{\sigma}$ is lightlike, we have that

$$(p(t), p(t)) = m\lambda(m\lambda|\mathfrak{q}|^2 + 2t(\sigma, \mathfrak{q})) = m\lambda^2(m - t)|\mathfrak{q}|^2.$$

Therefore, p admits non-zero roots and $\hat{f}$ is complementary surface of f with respect to p.

If $(p(0), p(0))$ is non-zero then

$$f \cap \langle p(0)\rangle^\perp = f \cap \hat{f} = \hat{f} \cap \langle p(0)\rangle^\perp.$$

Hence, f and $\hat{f}$ project to the same Guichard surface in the conformal geometry $\langle p(0)\rangle^\perp$. If $(p(0), p(0)) = 0$ then, by Corollary 6.20, $p(0)$ lies nowhere in f and we must have that $p(0) \in \Gamma\hat{f}$. Thus, $\hat{f}$ is totally umbilic.

We have thus arrived at the following theorem:

Theorem 6.34. *Suppose that $\hat{f}$ is a Darboux transform of f. Then there exists a constant vector $\mathfrak{q} \in \Gamma(f+\hat{f})$ if and only if f is a type one special Ω-surface that admits $\hat{f}$ as a complementary surface. Furthermore, if $\mathfrak{q}$ is lightlike then f projects to an L-isothermic surface in the Laguerre geometry defined by $\mathfrak{q}$ and $\hat{f}$ is totally umbilic. Otherwise, f and $\hat{f}$ project to the same Guichard surface in the conformal geometry $\langle \mathfrak{q} \rangle^{\perp}$.*

Remark 6.35. *Theorem 6.34 gives us a characterisation of L-isothermic surfaces and Guichard surfaces amongst Ω-surfaces in terms of their Darboux transforms.*

6.4 Type two special Ω-surfaces

In [33] Darboux introduced the classical notion of special isothermic surfaces in $\mathbb{R}^3$ and this was generalised to arbitrary three dimensional space forms in [1]. This notion was given a conformally invariant treatment in [18, 64] where it was shown that special isothermic surfaces in certain three dimensional space forms are those isothermic surfaces admitting a second order conserved quantity.

In [39] Eisenhart defines special Ω-surfaces and in the spirit of [18, 64] we shall show that these coincide with type two special Ω-surfaces.

Let $\mathfrak{p} \in \mathbb{R}^{4,2}$ be a point sphere complex. Suppose that $\hat{f}$ is an umbilic-free Darboux transformation of f with parameter m and assume that $s_0 = f \cap \hat{f}$ lies nowhere in $\langle \mathfrak{p} \rangle^{\perp}$. Let

$$\Lambda := f \cap \langle \mathfrak{p} \rangle^{\perp} \quad \text{and} \quad \hat{\Lambda} := \hat{f} \cap \langle \mathfrak{p} \rangle^{\perp}$$

and assume that these never become curvature spheres of f and $\hat{f}$, respectively. Then in accordance with [18, 64], we define the Ribaucour sphere congruence of Λ and $\hat{\Lambda}$ as the bundle of $(3,1)$-planes given by

$$V_R := \Lambda^{(1)} \oplus \hat{\Lambda} = \hat{\Lambda}^{(1)} \oplus \Lambda.$$

Now $V_R^{\perp} = s_0 \oplus \langle \mathfrak{p} \rangle$ and we define the cyclic system of Λ and $\hat{\Lambda}$ to be

$$\mathcal{C} := \Lambda \oplus \hat{\Lambda} \oplus (V_R^{\perp} \cap \langle \mathfrak{p} \rangle^{\perp}).$$

Furthermore, given a space form vector $\mathfrak{q} \in \langle \mathfrak{p} \rangle^{\perp}$ that never belongs to $\mathcal{C}$, we define the circle plane family of Λ and $\hat{\Lambda}$ with respect to $\mathfrak{q}$ to be

$$\mathcal{P} := \mathcal{C} \oplus \langle \mathfrak{q} \rangle.$$

Now since $V_R^{\perp} = s_0 \oplus \langle \mathfrak{p} \rangle$, we have that

$$\mathcal{W} := \mathcal{P} \oplus_{\perp} \langle \mathfrak{p} \rangle = (f + \hat{f}) \oplus \langle \mathfrak{q} \rangle \oplus \langle \mathfrak{p} \rangle.$$

We will refer to $\mathcal{W}$ as the circle plane family of f and $\hat{f}$ with respect to $\mathfrak{q}$ and $\mathfrak{p}$.

Eisenhart [39] defines a special Ω-surface to be an Ω-surface that admits two Darboux transforms with distinct parameters such that the circle planes of the transforms coincide. Therefore suppose that f admits two umbilic-free Darboux transforms $\hat{f}_1$ and $\hat{f}_2$ with respective distinct parameters m_1 and m_2 such that that their respective circle planes $\mathcal{P}_1$ and $\mathcal{P}_2$ with respect to a space form vector $\mathfrak{q} \in \langle \mathfrak{p} \rangle^\perp$ coincide. This is equivalent to

$$\mathcal{W}_1 := \mathcal{P}_1 \oplus_\perp \langle \mathfrak{p} \rangle = \mathcal{P}_2 \oplus_\perp \langle \mathfrak{p} \rangle =: \mathcal{W}_2.$$

We shall make the further assumption that

$$\mathcal{W}_2 \cap f_1 = f = \mathcal{W}_2 \cap f_2,$$

where f_1 and f_2 denote the derived bundles of f along the curvature subbundles T_1 and T_2, respectively. Now suppose that $\hat{\sigma}_i \in \Gamma \hat{f}_i$ is a parallel section of $d + m_i \eta^m$ for $i \in \{1, 2\}$. Then by the assumption that $\mathcal{W}_1 = \mathcal{W}_2$ we have that

$$\hat{\sigma}_1 = \sigma + \alpha \hat{\sigma}_2 + \beta \mathfrak{q} + \gamma \mathfrak{p},$$

for some $\sigma \in \Gamma f$ and smooth functions α, β and γ. Then the condition that $\hat{\sigma}_1$ is a parallel section of $d + m_1 \eta^m$ is equivalent to

$$0 = d\sigma + d\alpha\, \hat{\sigma}_2 + (m_1 - m_2)\eta^m \hat{\sigma}_2 + d\beta\, \mathfrak{q} + m_1 \beta \eta^m \mathfrak{q} + d\gamma\, \mathfrak{p} + m_1 \gamma \eta^m \mathfrak{p}.$$

Therefore, for any $X \in \Gamma T\Sigma$,

$$-d_X\sigma - (m_1 - m_2)\eta^m(X)\hat{\sigma}_2 - m_1 \eta^m(X)(\beta \mathfrak{q} + \gamma \mathfrak{p}) = d_X\alpha\, \hat{\sigma}_2 + d_X\beta\, \mathfrak{q} + d_X\gamma\, \mathfrak{p}. \quad (6.11)$$

Suppose that $X \in \Gamma T_i$. Then by Lemma 3.34, $\eta^m(X) \in \Gamma(f \wedge f_i)$ and the left hand side of (6.11) takes values in f_i, whilst the right hand side takes values in $\mathcal{W}_2$. Thus,

$$d_X\alpha\, \hat{\sigma}_2 + d_X\beta\, \mathfrak{q} + d_X\gamma\, \mathfrak{p} \in \Gamma(\mathcal{W}_2 \cap f_i) = \Gamma f,$$

by our assumption that $\mathcal{W}_2 \cap f_i = f$. Now, since $\mathfrak{q}$ never takes values in $(f + \hat{f}_2) \oplus \langle \mathfrak{p} \rangle$, we have that $d_X\beta = 0$. Furthermore, by our assumption that $f \cap \hat{f}_2$ lies nowhere in $\langle \mathfrak{p} \rangle^\perp$, we have that $\mathfrak{p}$ lies nowhere in $f + \hat{f}_2$ and thus $d_X\alpha = d_X\gamma = 0$. As our curvature direction X was arbitrary, we have that

$$d\alpha = d\beta = d\gamma = 0.$$

Now consider the polynomial $q(t) = q_0 + tq_1 + t^2 q_2$ such that

$$q(0) = \frac{m_1 m_2}{m_2 - m_1}(\beta\, \mathfrak{q} + \gamma\, \mathfrak{p}), \quad q(m_1) = m_1 \hat{\sigma}_1 \quad \text{and} \quad q(m_2) = \alpha m_2 \hat{\sigma}_2.$$

Then

$$\begin{aligned}
q_2 &= \frac{1}{m_2 m_1(m_1-m_2)}(m_2 q(m_1) - m_1 q(m_2) - (m_2-m_1)q(0)) \\
&= \frac{1}{m_1-m_2}(\hat{\sigma}_1 - \alpha\,\hat{\sigma}_2 - \beta\,\mathfrak{q} + \gamma\,\mathfrak{p}) \\
&= \frac{1}{m_1-m_2}\sigma.
\end{aligned}$$

Thus, $\eta q_2 = 0$ and

$$(d + t\eta^m)q(t)$$

is a quadratic polynomial with three zeroes, namely, 0, m_1 and m_2. Thus, it is the zero polynomial and q is a conserved quantity of the middle connection.

Now if $q_2 = 0$ then q is a linear conserved quantity of $d + t\eta^m$. Then by Proposition 6.4, $q_1 \in \Gamma f$. Thus,

$$\hat{\sigma}_1 - q(0) = \frac{1}{m_1}q(m_1) - q(0) \in \Gamma f.$$

This would imply that $\beta\,\mathfrak{q} + \gamma\,\mathfrak{p} \in \Gamma(f + \hat{f}_1)$. Since $\hat{\sigma}_1 \notin \Gamma\langle\hat{\sigma}_2\rangle$, we have that β and γ are not both zero. This would then contradict that

$$\mathcal{W}_1 = (f + \hat{f}_1) \oplus \langle\mathfrak{q}\rangle \oplus \langle\mathfrak{p}\rangle.$$

Hence, q is a degree two conserved quantity of $d + t\eta^m$ and f is a special Ω-surface of type two. We have thus proved:

Theorem 6.36. *Suppose that f projects to a special Ω-surface (in the sense of Eisenhart [39]) in a space form defined by $\mathfrak{q}$ and $\mathfrak{p}$. Then f is a special Ω-surface of type two whose degree two polynomial conserved quantity q satisfies $q(0) \in \langle\mathfrak{q},\mathfrak{p}\rangle$.*

Conversely, suppose that f is a special Ω-surface of type two with degree two conserved quantity

$$q(t) = q_0 + tq_1 + t^2 q_2.$$

Then $(q(t), q(t))$ is a polynomial with degree less than or equal to three. Suppose that m is a non-zero root of $(q(t), q(t))$ and let

$$\hat{f} := s_0 \oplus \langle q(m)\rangle,$$

where $s_0 := f \cap \langle q(m)\rangle^{\perp}$, be the corresponding complementary surface. By Proposition 6.4, $q_2 \in \Gamma f$. Thus,

$$q(m) = q_0 + mq_1 + m^2 q_2 \in \Gamma(f + \langle q_0, q_1\rangle).$$

Thus,

$$f + \hat{f} + \langle q_0\rangle = f + \langle q_0, q_1\rangle. \tag{6.12}$$

Now the right hand side of (6.12) does not depend upon m. Therefore, if $\hat{f}_1$ and $\hat{f}_2$ are complementary surfaces of f with respect to q then

$$f + \hat{f}_1 + \langle q_0 \rangle = f + \hat{f}_2 + \langle q_0 \rangle.$$

Therefore if we assume that q_0 never belongs to $f + \hat{f}_1$ or $f + \hat{f}_2$ then for any space form vector $\mathfrak{q}$ and point sphere complex $\mathfrak{p}$ such that $q_0 \in \langle \mathfrak{q}, \mathfrak{p} \rangle$ and

$$(f + \hat{f}_i) \cap \langle \mathfrak{q}, \mathfrak{p} \rangle = \emptyset$$

for $i \in \{1, 2\}$, we have that

$$(f + \hat{f}_1) \oplus \langle \mathfrak{q}, \mathfrak{p} \rangle = f \oplus \langle \mathfrak{q}, \mathfrak{p}, q_1 \rangle = (f + \hat{f}_2) \oplus \langle \mathfrak{q}, \mathfrak{p} \rangle.$$

We have thus arrived at the following theorem:

Theorem 6.37. *Suppose that f is a special Ω-surface of type two with degree two conserved quantity q. Then for any complementary surfaces $\hat{f}_1$ and $\hat{f}_2$ such that q_0 never belongs to $f + \hat{f}_1$ or $f + \hat{f}_2$ we have that the circle plane families of $\hat{f}_1$ and $\hat{f}_2$ coincide in any space form determined by space form vector $\mathfrak{q}$ and point sphere complex $\mathfrak{p}$ such that $q_0 \in \langle \mathfrak{q}, \mathfrak{p} \rangle$.*

6.4.1 Special isothermic and Guichard surfaces

Suppose that f is an Ω-surface whose middle connection admits a quadratic conserved quantity q and a linear conserved quantity p such that the polynomial $(p(t), p(t))$ is a non-zero constant (or, is linear with non-zero constant term) and $\mathfrak{q} := q(0)$ and $\mathfrak{p} := p(0)$ are linearly independent. Thus, f projects to an isothermic (respectively, Guichard surface) in the conformal geometry of $\langle \mathfrak{p} \rangle^{\perp}$. We may now define a third order conserved quantity r as follows:

$$r(t) := (p(t), p(t))q(t) - (q(t), p(t))p(t).$$

Then

$$(r(t), p(t)) = (p(t), p(t))(q(t), p(t)) - (p(t), q(t))(p(t), p(t)) = 0. \tag{6.13}$$

Now suppose that m is a non-zero root of $(r(t), r(t))$ and define $s_0 := f \cap \langle r(m) \rangle^{\perp}$. Then $\hat{f} := s_0 \oplus \langle r(m) \rangle$ is a Darboux transform of f with parameter m. Now $\hat{s} := \langle r(m) \rangle$ is a parallel subbundle of $d + m\eta^m$ and by (6.13), $(r(m), p(m)) = 0$. Thus, $\hat{f}$ is the Legendre lift of a Darboux transform (respectively, Eisenhart transformation) of $\Lambda := f \cap \langle \mathfrak{p} \rangle^{\perp}$, by Theorem 6.23 (respectively, Theorem 6.28). Furthermore, if

$$\langle \mathfrak{q}, \mathfrak{p} \rangle \cap (f + \hat{f}) = \emptyset,$$

then

$$(f + \hat{f}) \oplus \langle \mathfrak{q}, \mathfrak{p} \rangle = f \oplus \langle r(m), \mathfrak{q}, \mathfrak{p} \rangle = f \oplus \langle q_1, \mathfrak{q}, \mathfrak{p} \rangle. \tag{6.14}$$

Notice that the right hand side of (6.14) does not depend upon m. Therefore, let $\hat{f}_1$ and $\hat{f}_2$ be complementary surfaces of f with respect to r such that

$$\langle \mathfrak{q}, \mathfrak{p} \rangle \cap (f + \hat{f}_i) = \emptyset$$

for $i \in \{1, 2\}$. Then $\hat{f}_1$ and $\hat{f}_2$ are Legendre lifts of Darboux transforms (respectively, Eisenhart transformations) in $\langle \mathfrak{p} \rangle^\perp$ such that

$$(f + \hat{f}_1) \oplus \langle \mathfrak{q}, \mathfrak{p} \rangle = (f + \hat{f}_2) \oplus \langle \mathfrak{q}, \mathfrak{p} \rangle.$$

That is their circle planes families coincide in the space form $\mathfrak{Q}^3$ determined by space form vector $\mathfrak{q}$ and point sphere complex $\mathfrak{p}$. This implies that the projection of f into $\mathfrak{Q}^3$ is a special isothermic (respectively, special Guichard) surface according to [33] (respectively, [37]).

Conversely, suppose that f projects to an isothermic surface (Guichard surface) $\Lambda := f \cap \langle \mathfrak{p} \rangle^\perp$ in the conformal geometry $\langle \mathfrak{p} \rangle^\perp$. Then by Theorem 6.21 (respectively, Theorem 6.24), f is an Ω-surface whose middle connection admits a linear conserved quantity p with constant term $p(0) = \mathfrak{p}$. Now suppose that $\hat{f}_1$ and $\hat{f}_2$ project to Darboux transforms (respectively, Eisenhart transformations) of Λ in $\langle \mathfrak{p} \rangle^\perp$ with distinct parameters m_1 and m_2, respectively. Then, by Theorem 6.23 (respectively, Theorem 6.28), for $i \in \{1, 2\}$, the parallel subbundles $\hat{s}_i \leq \hat{f}_i$ of $d + m_i \eta^m$ satisfy $p(m_i) \in \Gamma \hat{s}_i^\perp$ (respectively, $p(-m_i) \in \Gamma \hat{s}_i^\perp$). If we further assume that for some space form vector $\mathfrak{q} \in \langle \mathfrak{p} \rangle^\perp$ that

$$(f + \hat{f}_1) \oplus \langle \mathfrak{q}, \mathfrak{p} \rangle = (f + \hat{f}_2) \oplus \langle \mathfrak{q}, \mathfrak{p} \rangle,$$

then by Theorem 6.36, f is a special Ω-surface of type two admitting a degree two polynomial conserved quantity q such that $q(0) \in \langle \mathfrak{q}, \mathfrak{p} \rangle$. We therefore have the following theorem:

Theorem 6.38. *Special isothermic (Guichard) surfaces in space forms are those Ω-surfaces that are simultaneously special Ω-surfaces of type one and type two.*

Chapter 7

Linear Weingarten surfaces

Let $\mathfrak{f} : \Sigma \to \mathfrak{Q}^3$ be the space form projection of a Legendre map $f : \Sigma \to \mathcal{Z}$ into the (Riemannian or Lorentzian) space form $\mathfrak{Q}^3$ with constant sectional curvature κ. Let χ be $+1$ in the case that $\mathfrak{Q}^3$ is a Riemannian space form and -1 in the case that it is Lorentzian. Recall the following definition:

Definition 7.1. *Where $\mathfrak{f}$ immerses we say that it is a linear Weingarten surface if*

$$aK + 2bH + c = 0 \tag{7.1}$$

for some $a, b, c \in \mathbb{R}$, not all zero, where $K = \kappa_1\kappa_2$ is the extrinsic Gauss curvature of $\mathfrak{f}$ and $H = \frac{1}{2}(\kappa_1 + \kappa_2)$ is the mean curvature of $\mathfrak{f}$.

A special case of linear Weingarten surfaces is given by flat fronts:

Definition 7.2. *A surface $\mathfrak{f} : \Sigma \to \mathfrak{Q}^3$ is a flat front if, where it immerses, the intrinsic Gauss curvature $K_{int} := \chi K + \kappa$ vanishes.*

In [14] it was shown that flat fronts in hyperbolic space are those Ω-surface whose isothermic sphere congruences each envelop a fixed sphere. In [15] it was shown that linear Weingarten surfaces in space forms corresponded to Lie applicable surfaces whose isothermic sphere congruences take values in certain linear sphere complexes. This theory was discretised in [16]. In this chapter we shall review this theory in terms of linear conserved quantities of the middle connection.

7.1 Parallel transformation

Recall from Section 2.4 how we break symmetry from Lie sphere geometry to space form geometry. Let $\mathfrak{q}, \mathfrak{p} \in \mathbb{R}^{4,2}$ be a space form vector and point sphere complex for a space form

$$\mathfrak{Q}^3 := \{y \in \mathcal{L} : (y, \mathfrak{q}) = -1, (y, \mathfrak{p}) = 0\}.$$

Now assume that $|\mathfrak{p}|^2 = \pm 1$. Then $\chi = -|\mathfrak{p}|^2$. and $\kappa = -|\mathfrak{q}|^2$.

Let $f : \Sigma \to Z$ be a Legendre map and assume that f projects into $\mathfrak{Q}^3$. Let $\mathfrak{f} : \Sigma \to \mathfrak{Q}^3$ denote the space form projection of f and let $\mathfrak{t} : \Sigma \to \mathfrak{P}^3$ denote its tangent plane congruence. Let

$$\begin{aligned}
\mathrm{ch}_{\mathfrak{q},\mathfrak{p}}(\phi) &:= \cosh\left(\tfrac{\sqrt{-|\mathfrak{q}|^2}}{\sqrt{|\mathfrak{p}|^2}}\phi\right),\\
\mathrm{sh}_{\mathfrak{q},\mathfrak{p}}(\phi) &:= \tfrac{\sqrt{|\mathfrak{p}|^2}}{\sqrt{-|\mathfrak{q}|^2}}\sinh\left(\tfrac{\sqrt{-|\mathfrak{q}|^2}}{\sqrt{|\mathfrak{p}|^2}}\phi\right),\\
\mathrm{th}_{\mathfrak{q},\mathfrak{p}}(\phi) &:= \tfrac{\mathrm{sh}_{\mathfrak{q},\mathfrak{p}}(\phi)}{\mathrm{ch}_{\mathfrak{q},\mathfrak{p}}(\phi)}.
\end{aligned}$$

Remark 7.3. *When $\mathfrak{q} \in \mathcal{L}$, we define $sh_{\mathfrak{q},\mathfrak{p}}(\phi)$ to be its continuous extension $sh_{\mathfrak{q},\mathfrak{p}}(\phi) = \phi$.*

A parallel transformation of the plane $\langle \mathfrak{q}, \mathfrak{p} \rangle$ is of the form

$$(\mathfrak{q}_\phi, \mathfrak{p}_\phi) = (\mathfrak{q}, \mathfrak{p}) \begin{pmatrix} \mathrm{ch}_{\mathfrak{q},\mathfrak{p}}(\phi) & \mathrm{sh}_{\mathfrak{q},\mathfrak{p}}(\phi) \\ -\frac{|\mathfrak{q}|^2}{|\mathfrak{p}|^2}\mathrm{sh}_{\mathfrak{q},\mathfrak{p}}(\phi) & \mathrm{ch}_{\mathfrak{q},\mathfrak{p}}(\phi) \end{pmatrix}. \tag{7.2}$$

Let

$$\mathfrak{Q}^3_\phi := \{y \in \mathcal{L} : (y, \mathfrak{q}_\phi) = -1, (y, \mathfrak{p}_\phi) = 0\}.$$

Then the projection, $\mathfrak{f}_\phi$, of f into $\mathfrak{Q}^3_\phi$ can be identified with a parallel surface of $\mathfrak{f}$. Now $\mathfrak{f}$ is immersed at a point $p \in \Sigma$ if and only if it doesn't coincide with a curvature sphere of f at p.

Proposition 7.4. *Suppose that $\mathfrak{f}$ is immersed at $p \in \Sigma$. The parallel surface $\mathfrak{f}_\phi$ corresponding to $(\mathfrak{q}_\phi, \mathfrak{p}_\phi)$ is immersed at p if and only if*

$$\frac{1}{th_{\mathfrak{q},\mathfrak{p}}(\phi)} \neq \kappa_i(p),$$

where κ_i, $i \in \{1, 2\}$, denote the principal curvatures of $\mathfrak{f}$.

Proof. The parallel surface $\mathfrak{f}_\phi$ is immersed at p if and only if it is not a curvature sphere at p. Now, for $i \in \{1, 2\}$, $\mathfrak{t} + \kappa_i \mathfrak{f}$ are lifts of the curvature spheres. Thus, $\mathfrak{f}_\phi$ is a curvature sphere at p if and only if at p

$$\begin{aligned}
0 &= (\mathfrak{t} + \kappa_i \mathfrak{f}, \mathfrak{p}_\phi)\\
&= (\mathfrak{t} + \kappa_i \mathfrak{f}, \mathrm{sh}_{\mathfrak{q},\mathfrak{p}}(\phi)\, \mathfrak{q} + \mathrm{ch}_{\mathfrak{q},\mathfrak{p}}(\phi)\, \mathfrak{p})\\
&= -\mathrm{ch}_{\mathfrak{q},\mathfrak{p}}(\phi) - \kappa_i \mathrm{sh}_{\mathfrak{q},\mathfrak{p}}(\phi),
\end{aligned}$$

for some $i \in \{1, 2\}$. $\square$

Remark 7.5. *Using Proposition 7.4, one can deduce the result of [59] that parallel surfaces fail to immerse at at most two values of ϕ.*

For $i \in \{1, 2\}$, let $\sigma_i = \mathfrak{t} + \kappa_i \mathfrak{f}$ be a lift of the curvature sphere s_i. Then, where a parallel surface $\mathfrak{f}_\phi$ is immersed, the principal curvatures are

$$\begin{aligned}\kappa_{i,\phi} = \frac{(\sigma_i, \mathfrak{q}_\phi)}{(\sigma_i, \mathfrak{p}_\phi)} &= \frac{\mathrm{ch}_{\mathfrak{q},\mathfrak{p}}(\phi)(\sigma_i, \mathfrak{q}) - \frac{|\mathfrak{q}|^2}{|\mathfrak{p}|^2}\mathrm{sh}_{\mathfrak{q},\mathfrak{p}}(\phi)(\sigma_i, \mathfrak{p})}{\mathrm{sh}_{\mathfrak{q},\mathfrak{p}}(\phi)(\sigma_i, \mathfrak{q}) + \mathrm{ch}_{\mathfrak{q},\mathfrak{p}}(\phi)(\sigma_i, \mathfrak{p})} \\ &= \frac{\kappa_i - \frac{|\mathfrak{q}|^2}{|\mathfrak{p}|^2}\mathrm{th}_{\mathfrak{q},\mathfrak{p}}(\phi)}{\mathrm{th}_{\mathfrak{q},\mathfrak{p}}(\phi)\kappa_i + 1},\end{aligned}$$

and the mean and Gauss curvature are given by

$$\begin{aligned} H_\phi &= \frac{1}{2}\frac{2\mathrm{th}_{\mathfrak{q},\mathfrak{p}}(\phi)K + 2(1 - \frac{|\mathfrak{q}|^2}{|\mathfrak{p}|^2}\mathrm{th}^2_{\mathfrak{q},\mathfrak{p}}(\phi))H - 2\frac{|\mathfrak{q}|^2}{|\mathfrak{p}|^2}\mathrm{th}_{\mathfrak{q},\mathfrak{p}}(\phi)}{\mathrm{th}^2_{\mathfrak{q},\mathfrak{p}}(\phi)K + 2\mathrm{th}_{\mathfrak{q},\mathfrak{p}}(\phi)H + 1} \\ K_\phi &= \frac{K - 2\frac{|\mathfrak{q}|^2}{|\mathfrak{p}|^2}\mathrm{th}_{\mathfrak{q},\mathfrak{p}}(\phi)H + (\frac{|\mathfrak{q}|^2}{|\mathfrak{p}|^2})^2\mathrm{th}^2_{\mathfrak{q},\mathfrak{p}}(\phi)}{\mathrm{th}^2_{\mathfrak{q},\mathfrak{p}}(\phi)K + 2\mathrm{th}_{\mathfrak{q},\mathfrak{p}}(\phi)H + 1}.\end{aligned}$$

Recalling that $\chi = -|\mathfrak{p}|^2$ and $\kappa = -|\mathfrak{q}|^2$, one can deduce the following theorem:

Theorem 7.6. *Away from points where $\chi K_\phi + \kappa \neq 0$ the function*

$$f\left(K_\phi, H_\phi, \frac{\kappa}{\chi}\right) := \frac{H_\phi^2 - K_\phi}{(K_\phi + \frac{\kappa}{\chi})^2}$$

is invariant under parallel transformation. If $\chi K_\phi + \kappa = 0$ then K_ϕ is invariant under parallel transformation.

7.2 Linear Weingarten surfaces

Similarly to [16], we have an alternative characterisation of the linear Weingarten condition:

Proposition 7.7. *$\mathfrak{f}$ is a linear Weingarten surface satisfying (7.1) if and only if*

$$[W](s_1, s_2) = [W(s_1, s_2)] = 0,$$

where $[W] \in \mathbb{P}(S^2\mathbb{R}^{4,2})$ is defined by

$$W := a\,\mathfrak{q} \odot \mathfrak{q} + 2b\,\mathfrak{q} \odot \mathfrak{p} + c\,\mathfrak{p} \odot \mathfrak{p},$$

and $s_1, s_2 \leq f$ are the curvature spheres of f.

Proof. Since W is a tensor, we may use the lifts

$$\sigma_1 = \mathfrak{t} + \kappa_1\mathfrak{f} \quad \text{and} \quad \sigma_2 = \mathfrak{t} + \kappa_2\mathfrak{f}$$

of the curvature spheres. Then,

$$\begin{aligned}
\mathfrak{q} \odot \mathfrak{q}(\sigma_1, \sigma_2) &= \frac{1}{2}((\sigma_1, \mathfrak{q})(\sigma_2, \mathfrak{q}) + (\sigma_2, \mathfrak{q})(\sigma_1, \mathfrak{q})) \\
&= \kappa_1 \kappa_2 \\
&= K,
\end{aligned}$$

$$\begin{aligned}
\mathfrak{q} \odot \mathfrak{p}(\sigma_1, \sigma_2) &= \frac{1}{2}((\sigma_1, \mathfrak{q})(\sigma_2, \mathfrak{p}) + (\sigma_2, \mathfrak{q})(\sigma_1, \mathfrak{p})) \\
&= \frac{1}{2}(\kappa_1 + \kappa_2) \\
&= H,
\end{aligned}$$

$$\begin{aligned}
\mathfrak{p} \odot \mathfrak{p}(\sigma_1, \sigma_2) &= \frac{1}{2}((\sigma_1, \mathfrak{p})(\sigma_2, \mathfrak{p}) + (\sigma_2, \mathfrak{p})(\sigma_1, \mathfrak{p})) \\
&= 1.
\end{aligned}$$

Therefore,

$$W(\sigma_1, \sigma_2) = aK + 2bH + c.$$

Hence, $aK + 2bH + c = 0$ if and only if $W(\sigma_1, \sigma_2) = 0$. $\square$

Now suppose that $(\mathfrak{q}_\phi, \mathfrak{p}_\phi)$ are given by a parallel transformation of $\langle \mathfrak{q}, \mathfrak{p} \rangle$ as in (7.2). Then we may write W in terms of $\mathfrak{q}_\phi$ and $\mathfrak{p}_\phi$:

$$W = \mathrm{ch}^2_{\mathfrak{q},\mathfrak{p}}(\phi)(a_\phi\, \mathfrak{q}_\phi \odot \mathfrak{q}_\phi + 2b_\phi\, \mathfrak{q}_\phi \odot \mathfrak{p}_\phi + c_\phi\, \mathfrak{p}_\phi \odot \mathfrak{p}_\phi),$$

where

$$\begin{aligned}
a_\phi &:= a - 2b\, \mathrm{th}_{\mathfrak{q},\mathfrak{p}}(\phi) + c\, \mathrm{th}^2_{\mathfrak{q},\mathfrak{p}}(\phi), \\
b_\phi &:= a\, \tfrac{|\mathfrak{q}|^2}{|\mathfrak{p}|^2} \mathrm{th}_{\mathfrak{q},\mathfrak{p}}(\phi) + b\, (1 - \tfrac{|\mathfrak{q}|^2}{|\mathfrak{p}|^2} \mathrm{th}^2_{\mathfrak{q},\mathfrak{p}}(\phi)) - c\, \mathrm{th}_{\mathfrak{q},\mathfrak{p}}(\phi), \\
c_\phi &:= a\, \tfrac{|\mathfrak{q}|^4}{|\mathfrak{p}|^4} \mathrm{th}^2_{\mathfrak{q},\mathfrak{p}}(\phi) + 2b\, \tfrac{|\mathfrak{q}|^2}{|\mathfrak{p}|^2} \mathrm{th}_{\mathfrak{q},\mathfrak{p}}(\phi) + c.
\end{aligned}$$

Since the condition $W(s_1, s_2) = 0$ doesn't depend upon space form projection, using Proposition 7.7 we obtain the following proposition:

Proposition 7.8. *Where it immerses, the parallel surface $\mathfrak{f}_\phi$ is a linear Weingarten surface satisfying*

$$a_\phi K_\phi + 2b_\phi H_\phi + c_\phi = 0.$$

Remark 7.9. *If we assume that $|\mathfrak{p}|^2 = -1$, i.e., $\mathfrak{Q}^3$ is a Riemannian space form, then Proposition 7.8 recovers a result proved in [50, Chapter 2].*

7.3 Linear Weingarten surfaces in Lie geometry

We shall now recover the results of [15] regarding the Lie applicability of umbilic-free linear Weingarten surfaces.

Theorem 7.10. *$\mathfrak{f}$ is an umbilic-free linear Weingarten surface satisfying (7.1) if and only if f is a Lie applicable surface with middle connection*

$$\eta^m = c\,\mathfrak{f}\wedge d\mathfrak{f} - b\,(\mathfrak{f}\wedge d\mathfrak{t} + \mathfrak{t}\wedge d\mathfrak{f}) + a\,\mathfrak{t}\wedge d\mathfrak{t}.$$

Furthermore, tubular linear Weingarten surfaces give rise to Ω_0-surfaces and non-tubular linear Weingarten surfaces give rise to Ω-surfaces whose isothermic sphere congruences are real in the case that $b^2-ac>0$ and complex conjugate in the case that $b^2-ac<0$.

Proof. Let

$$\eta := c\,\mathfrak{f}\wedge d\mathfrak{f} - b\,(\mathfrak{f}\wedge d\mathfrak{t} + \mathfrak{t}\wedge d\mathfrak{f}) + a\,\mathfrak{t}\wedge d\mathfrak{t}.$$

Then

$$\begin{aligned} d\eta &= c\,d\mathfrak{f}\curlywedge d\mathfrak{f} - b\,(d\mathfrak{f}\curlywedge d\mathfrak{t} + d\mathfrak{t}\curlywedge d\mathfrak{f}) + a\,d\mathfrak{t}\curlywedge d\mathfrak{t}\\ &= (a\,K + 2b\,H + c)\,d\mathfrak{f}\curlywedge d\mathfrak{f}. \end{aligned}$$

Thus η is closed if and only if $\mathfrak{f}$ is a linear Weingarten surface satisfying (7.1). Furthermore, one can check that

$$\eta = \tfrac{1}{\kappa_1-\kappa_2}((a\kappa_1+b)(\mathfrak{t}+\kappa_2\mathfrak{f})\wedge d(\mathfrak{t}+\kappa_2\mathfrak{f}) - (a\kappa_2+b)(\mathfrak{t}+\kappa_1\mathfrak{f})\wedge d(\mathfrak{t}+\kappa_1\mathfrak{f}))\, mod\,\Omega^1(\wedge^2 f).$$

Since $\mathfrak{t}+\kappa_1\mathfrak{f}\in\Gamma s_1$ and $\mathfrak{t}+\kappa_2\mathfrak{f}\in\Gamma s_2$, we have that the $\Omega^1(S_1\wedge S_2)$ part of η lies in $\Omega^1(\wedge^2 f)$. Thus η is the middle connection η^m.

Now the quadratic differential induced by η^m is given by

$$\begin{aligned} q &= -c(d\mathfrak{f},d\mathfrak{f}) + 2b(d\mathfrak{f},d\mathfrak{t}) - a(d\mathfrak{t},d\mathfrak{t})\\ &= (-c-2b\kappa_1-a\kappa_1^2)(d_1\mathfrak{f},d_1\mathfrak{f}) + (-c-2b\kappa_2-a\kappa_2^2)(d_2\mathfrak{f},d_2\mathfrak{f}), \end{aligned}$$

using Rodrigues' equations, $d_i\mathfrak{t}+\kappa_i d_i\mathfrak{f}=0$. Since $c=-b(\kappa_1+\kappa_2)-a\kappa_1\kappa_2$, we have that

$$q = (\kappa_1-\kappa_2)(-(a\kappa_1+b)(d_1\mathfrak{f},d_1\mathfrak{f}) + (a\kappa_2+b)(d_2\mathfrak{f},d_2\mathfrak{f})).$$

Since $\mathfrak{f}$ is an umbilic-free immersion, i.e., $\kappa_1\neq\kappa_2$, q is non-zero. Moreover,

$$-(a\kappa_1+b)(a\kappa_2+b) = -a^2K - 2abH - b^2 = -(b^2-ac).$$

Therefore, q is degenerate if and only if $b^2-ac=0$ if and only if $\mathfrak{f}$ is tubular. Furthermore, if $b^2-ac>0$ then q is indefinite and the isothermic sphere congruences of f are real, whereas if

$b^2 - ac < 0$ then q is positive definite and the isothermic sphere congruences of f are complex conjugate. $\square$

Corollary 7.11. *$\mathfrak{f}$ is a linear Weingarten surface satisfying (7.1) if and only if*

$$p(t) := \mathfrak{p} + t(-b\mathfrak{f} + a\mathfrak{t}) \quad \textit{and} \quad q(t) := \mathfrak{q} + t(c\mathfrak{f} - b\mathfrak{t})$$

are conserved quantities of the middle connection $d + t\eta^m$.

Proof. Suppose that $\mathfrak{f}$ is a linear Weingarten surface satisfying (7.1). Then it follows immediately from Theorem 7.10 that

$$\eta^m \mathfrak{p} = b\, d\mathfrak{f} - a\, d\mathfrak{t} \quad \text{and} \quad \eta^m \mathfrak{q} = -c\, d\mathfrak{f} + b\, d\mathfrak{t}.$$

Thus,

$$d(-b\,\mathfrak{f} + a\,\mathfrak{t}) + \eta^m \mathfrak{p} = 0 = d(c\,\mathfrak{f} - b\,\mathfrak{t}) + \eta^m \mathfrak{q}.$$

Hence, p and q are conserved quantities of $d + t\eta^m$.

Conversely, suppose that p and q are conserved quantities of $d + t\eta^m$. Thus

$$\eta^m \mathfrak{p} = b\, d\mathfrak{f} - a\, d\mathfrak{t} \quad \text{and} \quad \eta^m \mathfrak{q} = -c\, d\mathfrak{f} + b\, d\mathfrak{t}.$$

Then one can deduce that the middle connection has the form

$$\eta^m = c\,\mathfrak{f} \wedge d\mathfrak{f} - b\,(\mathfrak{f} \wedge d\mathfrak{t} + \mathfrak{t} \wedge d\mathfrak{f}) + a\,\mathfrak{t} \wedge d\mathfrak{t}.$$

Hence, by Theorem 7.10, $\mathfrak{f}$ is a linear Weingarten surface satisfying (7.1). $\square$

Clearly, real linear combinations of polynomial conserved quantities are polynomial conserved quantities. However, the degree of the polynomials may not be preserved. For example, one can check that there exists a constant conserved quantity within the span of the conserved quantities p and q of Corollary 7.11 if and only if $\mathfrak{f}$ is a tubular linear Weingarten surface, i.e., $b^2 - ac = 0$. Therefore, in the non-tubular case, any linear combination of p and q yields a linear conserved quantity of $d + t\eta^m$. In light of this we will consider two dimensional vector spaces of linear conserved quantities for Ω-surfaces:

7.3.1 Non-tubular linear Weingarten surfaces

Suppose that f is an Ω-surface and suppose that P is a two dimensional vector space of linear conserved quantities of $d + t\eta^m$. By $P(t)$ we shall denote the subbundle of $\underline{\mathbb{R}}^{4,2}$ formed by evaluating P at t.

Lemma 7.12. *For each $t \in \mathbb{R}$, $P(t)$ is a rank two subbundle of $\underline{\mathbb{R}}^{4,2}$.*

Proof. Let $p, q \in P$. Then by Lemma 6.19,

$$p(t) = \exp(t\,\sigma^+ \odot \sigma^-) p_0 \quad \text{and} \quad q(t) = \exp(t\,\sigma^+ \odot \sigma^-) q_0,$$

for some $q_0, p_0 \in \mathbb{R}^{4,2}$. Then $p(t)$ and $q(t)$ are linearly dependent sections of $P(t)$ for some $t \in \mathbb{R}$ if and only if p_0 and q_0 are linearly dependent if and only if p and q are linearly dependent. $\square$

We may equip P with a pencil of metrics $\{g_t\}_{t\in\mathbb{R}\cup\{\infty\}}$ defined for each $t \in \mathbb{R}$ and $\alpha, \beta \in P$ by

$$g_t(\alpha, \beta) := (\alpha(t), \beta(t)),$$

and

$$g_\infty := \lim_{t\to\infty} \tfrac{1}{t} g_t.$$

Thus, if we write $\alpha(t) = \alpha_0 + t\alpha_1$ and $\beta(t) = \beta_0 + t\beta_1$ then

$$g_\infty(\alpha, \beta) = (\alpha_0, \beta_1) + (\beta_0, \alpha_1),$$

Then, for general $t \in \mathbb{R}$, we have that

$$g_t = g_0 + t\, g_\infty.$$

We shall now consider the three dimensional vector space S^2P formed by the abstract symmetric product on P. For each $t \in \mathbb{R}$ we can identify elements of S^2P with symmetric endomorphisms on $\mathbb{R}^{4,2}$ via the map

$$\phi_t : S^2P \to S^2P(t), \quad \alpha \odot \beta \mapsto \alpha(t) \odot \beta(t).$$

Furthermore, we have an isomorphism from S^2P to the space of symmetric tensors on P with respect to g_∞, denoted $S^2_\infty P$ defined by

$$\phi_\infty : S^2P \to S^2_\infty P, \quad \alpha \odot \beta \mapsto (\alpha \odot \beta)_\infty,$$

where for $\gamma, \delta \in P$,

$$(\alpha \odot \beta)_\infty(\gamma, \delta) := \tfrac{1}{2}(g_\infty(\alpha, \gamma) g_\infty(\beta, \delta) + g_\infty(\alpha, \delta) g_\infty(\beta, \gamma)).$$

Using Corollary 7.11, we obtain the following theorem:

Theorem 7.13. *Suppose that $\mathfrak{f}$ is a non-tubular linear Weingarten surface satisfying (7.1). Then f is an Ω-surface whose middle connection admits a two dimensional space of linear conserved quantities P such that $g_0 \neq 0$ and g_∞ is non-degenerate. Furthermore, the linear Weingarten condition $[W]$ is given by $[\phi_0 \circ \phi_\infty^{-1}(g_\infty)]$.*

Proof. By Theorem 7.10, f is an Ω-surface and by Corollary 7.11, $P := \langle p, q\rangle$ is a two dimensional space of linear conserved quantities for $d + t\eta^m$, where

$$p(t) := \mathfrak{p} + t(-b\mathfrak{f} + a\mathfrak{t}) \quad \text{and} \quad q(t) := \mathfrak{q} + t(c\mathfrak{f} - b\mathfrak{t}).$$

Since $\mathfrak{p}$ is a point sphere complex, i.e., $|\mathfrak{p}|^2 \neq 0$, we have that $g_0 \neq 0$. We also have that

$$\Delta := g_\infty(p,p)g_\infty(q,q) - g_\infty(q,p)^2 = -4(b^2 - ac).$$

Therefore g_∞ is non-degenerate and

$$\phi_\infty^{-1} g_\infty = \Delta^{-1}(g_\infty(p,p) q \odot q - 2\, g_\infty(p,q) q \odot p + g_\infty(q,q) p \odot p).$$

Thus,

$$\begin{aligned} \phi_0(\phi_\infty^{-1} g_\infty) &= \Delta^{-1}(g_\infty(p,p)q(0) \odot q(0) - 2\, g_\infty(p,q) q(0) \odot p(0) + g_\infty(q,q) p(0) \odot p(0)) \\ &= \Delta^{-1}(-2a\, \mathfrak{q} \odot \mathfrak{q} - 4b\, \mathfrak{q} \odot \mathfrak{p} - 2c\, \mathfrak{p} \odot \mathfrak{p}) \\ &= -2\Delta^{-1}(a\, \mathfrak{q} \odot \mathfrak{q} + 2b\, \mathfrak{q} \odot \mathfrak{p} + c\, \mathfrak{p} \odot \mathfrak{p}). \end{aligned}$$

Hence, $[W] = [\phi_0(\phi_\infty^{-1} g_\infty)]$. □

Remark 7.14. *It follows from the proof of Theorem 7.13 that if $b^2 - ac > 0$ then g_∞ is indefinite and if $b^2 - ac < 0$ then g_∞ is definite. Then it follows by Theorem 7.10 that the isothermic sphere congruences are real when g_∞ is indefinite and complex conjugate when g_∞ is definite.*

We now seek a converse to Theorem 7.13, but first we will use the following technical lemma:

Lemma 7.15. *Suppose that $\mathfrak{q}, \mathfrak{p} \in P(0)$ are a space form vector and point sphere complex for a space form $\mathfrak{Q}^3$. Then f defines a point sphere map $\mathfrak{f} : \Sigma \to \mathfrak{Q}^3$ with tangent plane congruence $\mathfrak{t} : \Sigma \to \mathfrak{P}^3$ if and only if g_∞ is non-degenerate.*

Proof. Let $p, q \in P$ such that

$$p(t) = \exp(t\,\sigma^+ \odot \sigma^-)\mathfrak{p} \quad \text{and} \quad q(t) = \exp(t\,\sigma^+ \odot \sigma^-)\mathfrak{q}.$$

Then

$$\begin{aligned} g_\infty(p,p) &= 2(\sigma^+, \mathfrak{p})(\sigma^-, \mathfrak{p}), \\ g_\infty(q,q) &= 2(\sigma^+, \mathfrak{q})(\sigma^-, \mathfrak{q}), \quad \text{and} \\ g_\infty(p,q) &= (\sigma^+, \mathfrak{p})(\sigma^-, \mathfrak{q}) + (\sigma^+, \mathfrak{q})(\sigma^-, \mathfrak{p}). \end{aligned}$$

One can then deduce that

$$\begin{aligned} g_\infty(p,p)g_\infty(q,q) - g_\infty(p,q)^2 &= -((\sigma^+, \mathfrak{p})(\sigma^-, \mathfrak{q}) - (\sigma^+, \mathfrak{q})(\sigma^-, \mathfrak{p}))^2 \\ &= -((\sigma^+ \wedge \sigma^-)\mathfrak{p}, \mathfrak{q})^2. \end{aligned}$$

By Corollary 6.20, f lies nowhere in $\langle \mathfrak{q} \rangle^\perp$ or $\langle \mathfrak{p} \rangle^\perp$. It then follows by Lemma 2.27 that f defines a point sphere map $\mathfrak{f}$ and tangent plane congruence $\mathfrak{t}$ if and only if g_∞ is non-degenerate. □

We are now in a position to state the following theorem:

Theorem 7.16. *Suppose that f is an umbilic-free Ω-surface whose middle connection admits a two dimensional space of linear conserved quantities P, such that $g_0 \neq 0$ and g_∞ is non-degenerate. Then f projects to a non-tubular linear Weingarten surface with*

$$[W] = [\phi_0 \circ \phi_\infty^{-1}(g_\infty)],$$

where it immerses, in any space form determined by space form vector and point sphere complex $\mathfrak{q}, \mathfrak{p} \in P(0)$.

Proof. Since $g_0 \neq 0$ we may choose a space form vector $\mathfrak{q}$ and point sphere complex $\mathfrak{p}$ for a space form $\mathfrak{Q}^3$ from $P(0)$. By Lemma 7.15, since g_∞ is non-degenerate, f projects to a point sphere map $\mathfrak{f} : \Sigma \to \mathfrak{Q}^3$ with tangent plane congruence $\mathfrak{t} : \Sigma \to \mathfrak{P}^3$.

Now we may choose $p, q \in P$ such that $p(0) = \mathfrak{p}$ and $q(0) = \mathfrak{q}$. By Corollary 6.7, for certain Christoffel dual lifts $\sigma^\pm$, $(\sigma^\pm, \mathfrak{q})$ and $(\sigma^\pm, \mathfrak{p})$ are constant and

$$p(t) = \mathfrak{p} + t(\sigma^+ \odot \sigma^-)\mathfrak{p} \quad \text{and} \quad q(t) = \mathfrak{q} + t(\sigma^+ \odot \sigma^-)\mathfrak{q}.$$

Therefore, there exists constants (possibly complex) $\lambda^\pm$ and $\mu^\pm$ such that

$$\sigma^\pm = \lambda^\pm \mathfrak{f} + \mu^\pm \mathfrak{t}$$

and

$$\begin{aligned} p(t) &= \mathfrak{p} + t(-\tfrac{1}{2}(\mu^+\lambda^- + \mu^-\lambda^+)\mathfrak{f} - \mu^+\mu^-\mathfrak{t}) \quad \text{and} \\ q(t) &= \mathfrak{q} + t(-\lambda^+\lambda^-\mathfrak{f} - \tfrac{1}{2}(\mu^+\lambda^- + \mu^-\lambda^+)\mathfrak{t}). \end{aligned}$$

Then, by Corollary 7.11, where it immerses, $\mathfrak{f}$ is a linear Weingarten surface satisfying (7.1) with

$$a := -\mu^+\mu^-, \quad b := \frac{1}{2}(\mu^+\lambda^- + \mu^-\lambda^+) \quad \text{and} \quad c := -\lambda^+\lambda^-.$$

On the other hand

$$a = -\tfrac{1}{2}(p,p)_\infty, \quad b := \tfrac{1}{2}(p,q)_\infty \quad \text{and} \quad c := -\tfrac{1}{2}(q,q)_\infty.$$

Thus,

$$\begin{aligned} W &= a\,\mathfrak{q} \odot \mathfrak{q} + 2b\,\mathfrak{q} \odot \mathfrak{p} + c\,\mathfrak{p} \odot \mathfrak{p} \\ &= -\tfrac{1}{2}((p,p)_\infty q(0) \odot q(0) - 2(p,q)_\infty q(0) \odot p(0) + (q,q)_\infty p(0) \odot p(0)) \\ &= -\tfrac{\Delta}{2}(\phi_0(\phi_\infty^{-1} g_\infty)), \end{aligned}$$

where $\Delta := (p,p)_\infty (q,q)_\infty - (p,q)_\infty^2$. Furthermore, $b^2 - ac = \Delta$. Hence, $\mathfrak{f}$ is non-tubular. $\square$

If g_∞ is non-degenerate on P then g_∞ induces two null directions on P. In the case that g_∞ is indefinite these are real directions and in the case that g_∞ is definite they are complex conjugate. Let $q^\pm$ be two linearly independent vectors in $P \otimes \mathbb{C}$ and define $\mathfrak{q}^\pm := q^\pm(0) \in \mathbb{R}^{4,2} \otimes \mathbb{C}$. Then

$$q^\pm(t) = \exp(t\,\sigma^+ \odot \sigma^-)\mathfrak{q}^\pm.$$

Thus

$$(q^\pm, q^\pm)_\infty = 2(\sigma^\pm, \mathfrak{q}^\pm)(\sigma^\mp, \mathfrak{q}^\pm)$$

and

$$(q^+, q^-)_\infty = (\sigma^+, \mathfrak{q}^+)(\sigma^-, \mathfrak{q}^-) + (\sigma^+, \mathfrak{q}^-)(\sigma^-, \mathfrak{q}^+).$$

Therefore, $q^\pm$ are null with respect to $(\,,\,)_\infty$ if and only if we have (after possibly switching $q^\pm$) that $(\sigma^\pm, \mathfrak{q}^\pm) = 0$, i.e., the isothermic sphere congruences $s^\pm$ take values in $\langle \mathfrak{q}^\pm \rangle^\perp$. Now by applying Theorem 7.13 and Theorem 7.16 we obtain the main result of [15]:

Theorem 7.17. *Non-tubular linear Weingarten surfaces in space forms are those Ω-surfaces whose isothermic sphere congruences each take values in a linear sphere complex.*

Furthermore, by scaling $q^\pm$ appropriately we have that $g_\infty = (q^+ \odot q^-)_\infty$. Therefore, we have that

$$[W] = [\mathfrak{q}^+ \odot \mathfrak{q}^-],$$

which was shown in [16] for the discrete case.

7.3.2 Tubular linear Weingarten surfaces

In [15], the following theorem is proved:

Theorem 7.18. *Tubular linear Weingarten surfaces in space forms are those Ω_0-surfaces whose isothermic curvature sphere congruence takes values in a linear sphere complex.*

We shall recover this result in terms of our setup. Suppose that $\mathfrak{f}$ is a tubular linear Weingarten surface satisfying (7.1), i.e., $b^2 - ac = 0$. Then by Theorem 7.10, f is an Ω_0-surface and by Corollary 7.11 the middle connection $d + t\eta^m$ of f admits conserved quantities

$$p(t) := \mathfrak{p} + t(-b\mathfrak{f} + a\mathfrak{t}) \quad \text{and} \quad q(t) := \mathfrak{q} + t(c\mathfrak{f} - b\mathfrak{t}).$$

Then

$$\mathfrak{q}_0 := c\,p(t) + b\,q(t) = c\,\mathfrak{p} + b\,\mathfrak{q} + t(ac - b^2)\mathfrak{t} = c\,\mathfrak{p} + b\,\mathfrak{q}$$

is a non-zero constant conserved quantity of $d + t\eta^m$. This implies that $\eta^m \mathfrak{q}_0 = 0$. Without loss of generality, assume that the middle connection has the form

$$\eta^m = \sigma_1 \wedge \star d\sigma_1.$$

Then

$$0 = \eta^m \mathfrak{q}_0 = (\sigma_1, \mathfrak{q}_0) \star d\sigma_1 - (\star d\sigma_1, \mathfrak{q}_0)\sigma_1.$$

Since f is umbilic-free we have that $d_2\sigma_1$ does not take values in f and thus $(\sigma_1, \mathfrak{q}_0) = 0$, i.e., $s_1 \leq \langle \mathfrak{q}_0 \rangle^\perp$.

Conversely, suppose that f is an umbilic-free Legendre map such that $s_1 \leq \langle \mathfrak{q}_0 \rangle^\perp$. Let $\tilde{\mathfrak{q}}_0 \in \mathbb{R}^{4,2}$ such that the plane $\langle \mathfrak{q}_0, \tilde{\mathfrak{q}}_0 \rangle$ is not totally degenerate. Then let $[W] \in \mathbb{P}(S^2\mathbb{R}^{4,2})$ be defined by

$$W = \mathfrak{q}_0 \odot \mathfrak{q}_0.$$

Then since $s_1 \leq \langle \mathfrak{q}_0 \rangle^\perp$ we have that

$$[W](s_1, s_2) = 0.$$

Hence, by Proposition 7.7, away from points where $f \perp \langle \mathfrak{q}_0 \rangle$, f projects to a linear Weingarten surface, where it immerses, in any space form determined by space form vector and point sphere complex chosen from $\langle \mathfrak{q}_0, \tilde{\mathfrak{q}}_0 \rangle$. Furthermore, since the discriminant of W vanishes, such linear Weingarten surfaces are tubular.

Remark 7.19. *Since we assumed that f is umbilic-free, we have that $f \not\perp \langle \mathfrak{q}_0 \rangle$ on a dense open subset of Σ, by Lemma 2.23.*

Remark 7.20. *Notice in the converse argument to Theorem 7.18 that we did not have to assume that f was an Ω_0-surface. We can thus deduce that if one of the curvature sphere congruences of a Legendre map takes values in a linear sphere complex then it must be isothermic.*

7.4 Transformations of linear Weingarten surfaces

Using the identification of non-tubular linear Weingarten surfaces as certain Ω surfaces, we will apply the transformations of Chapter 5 to obtain new linear Weingarten surfaces.

Let f be an Ω-surface whose middle connection $d + t\eta^m$ admits a two dimensional space of linear conserved quantities P such that the $g_0 \neq 0$ and g_∞ is non-degenerate. Then, by Theorem 7.16, f projects to linear Weingarten surfaces with linear Weingarten condition

$$[W] = [\phi_0 \circ \phi_\infty^{-1}(g_\infty)],$$

in any space form determined by space form vector and point sphere complex chosen from $P(0)$.

7.4.1 Calapso transformations

In [15], the Calapso transformation for Ω-surfaces was used to obtain a Lawson correspondence for linear Weingarten surfaces. This was further investigated in [16] in the discrete setting. We shall recover this analysis in terms of linear conserved quantities of the middle connection.

Let $t \in \mathbb{R}$ and consider the Calapso transform $f^t = T(t)f$ of f. For each $p \in P$ we have by Proposition 6.14 that p^t defined by $p^t(s) = T(t)p(t+s)$ is a linear conserved quantity of the middle connection of f^t. Therefore, the middle connection of f^t admits a two dimensional space of linear conserved quantities P^t defined by the isomorphism

$$\Psi : P \to P^t, \quad p \mapsto p^t.$$

As with P, we may equip P^t with a pencil of metrics $\{g^t_s\}_{s\in\mathbb{R}\cup\{\infty\}}$. Then for each $s \in \mathbb{R}$ and $\alpha^t, \beta^t \in P$,

$$g^t_s(\alpha^t, \beta^t) = (T(t)\alpha(t+s), T(t)\beta(t+s)) = (\alpha(t+s), \beta(t+s)) = g_{t+s}(\alpha, \beta),$$

by the orthogonality of $T(t)$. Thus, Ψ is an isometry from (P, g_{t+s}) to (P^t, g^t_s). It is then clear that Ψ is an isometry from (P, g_∞) to (P^t, g^t_∞). Therefore, g^t_∞ is non-degenerate, and $g^t_0 \neq 0$ if and only if $g_t \neq 0$.

Proposition 7.21. *There exists $t \in \mathbb{R}^\times$ such that $g_t = 0$ if and only if f projects to a flat front in any space form determined by $P(0)$.*

Proof. Since

$$g_t = g_0 + t g_\infty,$$

for each $t \in \mathbb{R}^\times$, we have that $g_t = 0$ if and only if $g_0 = -t g_\infty$. Now let $q, p \in P$ be an orthogonal basis with respect to g_∞. Then $[W]$ is given by

$$W = \phi_0 \circ \phi_\infty^{-1}(g_\infty) = \frac{1}{g_\infty(q,q)} q(0) \odot q(0) + \frac{1}{g_\infty(p,p)} p(0) \odot p(0).$$

Thus, if $g_0 = -t g_\infty$ then $q(0)$ and $p(0)$ are orthogonal and define a space form vector and point sphere complex for a space form with sectional curvature $\kappa = -g_0(q,q)$ and assuming that p is normalised such that $g_0(p,p) = \pm 1$, $\chi = -g_0(p,p)$. Furthermore, by Proposition 7.7, f projects to a surface $\mathfrak{f}$ with constant extrinsic Gauss curvature

$$K = -\frac{g_\infty(q,q)}{g_\infty(p,p)} = -\frac{g_0(q,q)}{g_0(p,p)} = -\frac{\kappa}{\chi}$$

in this space form, i.e., $\mathfrak{f}$ is a flat front.

Conversely, suppose f projects to a flat front $\mathfrak{f}$ in a space form defined by space form vector $\mathfrak{q}$ and point sphere complex $\mathfrak{p}$, i.e., $\mathfrak{f}$ satisfies

$$\chi K + \kappa = 0.$$

Since $\kappa = -|\mathfrak{q}|^2$ and $\chi = -|\mathfrak{p}|^2$, by Corollary 7.11 we have that

$$p(t) = \mathfrak{p} + t\,|\mathfrak{p}|^2 \mathfrak{t} \quad \text{and} \quad q(t) = \mathfrak{q} + t\,|\mathfrak{q}|^2 \mathfrak{f}$$

are linear conserved quantities of the middle connection. Moreover,

$$g_{\frac{1}{2}}(p,p) = g_{\frac{1}{2}}(q,p) = g_{\frac{1}{2}}(q,q) = 0.$$

Hence, $g_{\frac{1}{2}} = 0$. $\square$

Now consider the maps

$$\phi_s^t : S^2 P^t \to S^2 P^t(s), \quad \alpha^t \odot \beta^t \mapsto \alpha^t(s) \odot \beta^t(s).$$

Then, by extending the action of Ψ to $S^2 P$ and $T(t)$ to $S^2 \underline{\mathbb{R}}^{4,2}$ in the standard way, one has that

$$\phi_s^t = T(t) \circ \phi_{t+s} \circ \Psi^{-1}.$$

Furthermore, if we define $\phi_\infty^t : S^2 P^t \to S^2_\infty P^t$ analogously to ϕ_∞, then as g_∞^t is isometric to g_∞ via Ψ, we have that

$$(\phi_\infty^t)^{-1} g_\infty^t = \Psi \circ \phi_\infty^{-1} g_\infty.$$

Applying Theorem 7.16, we have proved the following theorem:

Theorem 7.22. *Suppose that $g_t \neq 0$. Then f^t projects to a linear Weingarten surface with linear Weingarten condition*

$$[W^t] = [T(t)\phi_t(\phi_\infty^{-1} g_\infty)],$$

in any space form determined by space form vector and point sphere complex chosen from $P^t(0) = T(t)P(t)$.

In a similar way to [14], we have the following result regarding Calapso transforms of flat fronts:

Corollary 7.23. *Suppose that f projects to a flat front and let $t \in \mathbb{R}$ such that $g_t \neq 0$. Then f^t projects to a flat front in any space form determined by space form vector and point sphere complex chosen from $P^t(0) = T(t)P(t)$.*

Proof. Recall that for any $s \in \mathbb{R}$, g_s^t is isometric to g_{t+s} via Ψ. Now by Proposition 7.21, there exists $t_0 \in \mathbb{R}^\times$ such that $g_{t_0} = 0$. Therefore, $g_{t_0 - t}^t = 0$ and it follows by Proposition 7.21 that f^t projects to a flat front in $P^t(0)$. $\square$

7.4.2 Darboux transformations

Suppose that $\hat{f}$ is an umbilic-free Darboux transform of f with parameter m. Let $\hat{s} \leq \hat{f}$ be the parallel subbundle of $d + m\eta^m$ and let $s \leq f$ be the parallel subbundle of $d + t\hat{\eta}_m$. By Proposition 6.16, if p is a linear conserved quantity of $d + t\eta^m$ and $p(m) \in \Gamma\hat{s}^\perp$, then $\hat{p}$ is a linear conserved quantity of $d + t\hat{\eta}_m$ where

$$\hat{p}(t) = \Gamma_s^{\hat{s}}(1 - t/m)p(t).$$

Furthermore, $\hat{p}(0) = p(0)$ and $(\hat{p}(t), \hat{p}(t)) = (p(t), p(t))$. Now, if we assume that $P(m) \leq \hat{s}^\perp$, then $\hat{P}$ is a two dimensional space of linear conserved quantities for the middle connection of

$\hat{f}$, where we define $\hat{P}$ via the isomorphism

$$\Upsilon : P \to \hat{P}, \quad p \mapsto \hat{p}.$$

For each $t \in \mathbb{R}$ we shall let Υ_t denote the induced isomorphism between the subbundles $P(t)$ to $\hat{P}(t)$. Then $\hat{P}(0) = P(0)$ and $\Upsilon_0 = id_{P(0)}$. Furthermore, if we let $\{\hat{g}_t\}_{t\in\mathbb{R}\cup\{\infty\}}$ denote the pencil of metrics on $\hat{P}$, then, as $(\hat{p}(t), \hat{p}(t)) = (p(t), p(t))$, we have that (P, g_t) is isometric to $(\hat{P}, \hat{g}_t)$ via Υ for all $t \in \mathbb{R} \cup \{\infty\}$. In particular, we have that $\hat{g}_0 \neq 0$ and $\hat{g}_\infty$ is non-degenerate. Therefore, by Theorem 7.16, $\hat{f}$ projects to linear Weingarten surfaces in any space form defined by space form vector and point sphere complex chosen from $\hat{P}(0) = P(0)$.

As for f, define $\hat{\phi}_t : S^2\hat{P} \to S^2\hat{P}(t)$ and $\hat{\phi}_\infty : S^2\hat{P} \to S^2_\infty\hat{P}$ accordingly. Then for each $t \in \mathbb{R}$,

$$\hat{\phi}_t = \Upsilon_t \circ \phi_t \circ \Upsilon^{-1}$$

and $\hat{\phi}_0 = \phi_0$. Furthermore,

$$\hat{\phi}_\infty^{-1}(\hat{g}_\infty) = \phi_\infty^{-1}(g_\infty).$$

Then the linear Weingarten condition for $\hat{f}$ is given by

$$[\hat{W}] = [\hat{\phi}_0 \circ \hat{\phi}_\infty^{-1}(\hat{g}_\infty)] = [\phi_0 \circ \phi_\infty^{-1}(g_\infty)].$$

Therefore, we have proved the following theorem:

Theorem 7.24. *$\hat{f}$ is a linear Weingarten surface with the same linear Weingarten condition as f in any space form determined by space form vector and point sphere complex chosen from $P(0)$.*

Remark 7.25. *For each $m \in \mathbb{R}^\times$, there exists a two-parameter family of Darboux transforms such that $P(m) \leq s^\perp$. Therefore, given a linear Weingarten surface we may obtain a three parameter family of linear Weingarten surfaces satisfying the same linear Weingarten condition.*

Chapter 8

Weierstrass representations and period problems

The final chapter of this thesis is taken from a paper [46] written in collaboration with S. Fujimori and S. Gaber. Credit is also due to W. Rossman who provided extensive assistance and inspiration in the paper's creation.

With the recent interest in finding Weierstrass-type representations for surfaces other than minimal surfaces in Euclidean 3-space, the case of (spacelike) constant mean curvature (CMC) 1 surfaces in de-Sitter 3-space $\mathbb{S}^{2,1}$ has been undergoing investigation. (Throughout this chapter, we treat only spacelike surfaces with singularities.) Having more methods available for producing surfaces of this type is useful, and is the goal of this chapter.

In the recent work by Fujimori, Rossman, Umehara, Yamada and Yang [47], the method by Rossman, Umehara and Yamada in [62] was adapted to the case of maximal surfaces (surfaces with vanishing mean curvature) in Minkowski 3-space $\mathbb{R}^{2,1}$ and their cousin CMC 1 surfaces in de Sitter 3-space, for the purpose of producing some specific examples of surfaces with particular geometric properties of interest. Here we reformulate that result in [47] to apply to other surfaces as well, using a non-degeneracy condition like that used in [62], see Theorem 8.4.

Although in Euclidean 3-space $\mathbb{R}^3$ every direction is geometrically the same, this is not the case in $\mathbb{R}^{2,1}$. For this reason, when we formulate the non-degeneracy condition in Section 3, we use only two timelike planes in general position, rather than the three planes in general position that were used in [62].

In section 4 we give new examples of genus 1 maxfaces (maximal surfaces with certain admissible singularities, see for example [72]) in $\mathbb{R}^{2,1}$ with two or three ends and apply Theorem 8.4 to produce corresponding genus 1 CMC 1 faces (CMC 1 surfaces with certain admissible singularities, see [45]) in $\mathbb{S}^{2,1}$. With the final two examples, we are able to provide an answer to Problem 2 raised in [47]. In fact, one of those two examples has all ends embedded.

Remark 8.1. *In [15, Section 4] it was shown that linear Weingarten surfaces that possess a Weierstrass-type representation (for example, maximal surfaces in $\mathbb{R}^{2,1}$ and CMC 1 surfaces* *in $\mathbb{S}^{2,1}$) are L-isothermic surfaces (in an appropriate space form). Theorem 6.34 characterises L-isothermic surfaces as those surfaces that admit a totally umbilic Darboux transform. The author believes that this characterisation will unify all Weierstrass-type representations and yield a geometric interpretation for these representations.*

8.1 CMC surfaces in de Sitter 3-space

Let $\mathbb{R}^{3,1}$ be Minkowski 4-space with the metric of signature $(-,+,+,+)$. We define de Sitter space of constant sectional curvature 1 by

$$\mathbb{S}^{2,1} = \{(t,x,y,z) \in \mathbb{R}^{3,1} \mid -t^2 + x^2 + y^2 + z^2 = 1\} .$$

We will use the following standard 2×2-matrix model of $\mathbb{S}^{2,1}$ (see for example [FRUYY]):

$$\mathbb{S}^{2,1} = \{X e_3 X^* | X \in \mathrm{SL}(2,\mathbb{C})\} = \mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(1,1),$$

where

$$e_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

In this model the metric on $\mathbb{S}^{2,1}$ is determined by

$$\langle Y, Y \rangle = -\det(Y),$$

for $Y \in T_p\mathbb{S}^{2,1}$.

Let Σ be a Riemann surface. For $c \in \mathbb{R}^\times$, let $F_c : \Sigma \to \mathrm{SL}(2,\mathbb{C})$ be a solution of

$$dF_c \cdot F_c^{-1} = c \begin{pmatrix} G & -G^2 \\ 1 & -G \end{pmatrix} \omega \ , \tag{8.1}$$

where $G : \Sigma \to \mathbb{C}$ is a holomorphic function and $\omega \in \Omega^1(\Sigma)$ is a holomorphic 1-form independent of c. Then using the Weierstrass-type representation for $\mathbb{S}^{2,1}$ (see for example [47]) we have that

$$f_c := F_c e_3 F_c^*,$$

is a CMC 1 surface in $\mathbb{S}^{2,1}$.

Remark 8.2. *We have that $\omega = \frac{Q}{dG}$, where cQ is the Hopf differential of f_c.*

8.2 The method of Rossman, Umehara and Yamada in de-Sitter 3-space

Let D be a simply-connected region in Σ with local coordinate z, bounded by a finite number of smooth arcs. Let G and ω be Weierstrass data on D producing a (possibly branched) maxface

[72]

$$f_0 = \mathrm{Re} \int_{z_0}^{z} (-2G, 1+G^2, i - iG^2)\omega$$

in $\mathbb{R}^{2,1}$ (with metric of signature $(-,+,+)$) bounded by planar geodesic curvature lines lying in either of two given timelike planes P_1 and P_2, and suppose that P_1 and P_2 are not parallel to each other. We allow that these geodesics might be defined only in the interiors of the corresponding smooth arcs in ∂D, creating the possibility of ends of f_0 at the endpoints of the smooth arcs. The Hopf differential $Q = \omega dG$ is real when restricted to ∂D. Suppose that repeated inclusion of reflected copies of f_0 across P_1 and P_2 (and their images under reflections) extends f_0 to a (possibly branched) maxface $\hat{f}_0$ of finite topology and no boundary.

Label the smooth boundary arcs of D as $S_{1,1}$, $S_{1,2}$, ..., S_{1,k_1}, $S_{2,1}$, $S_{2,2}$, ..., S_{2,k_2}, where each $S_{i,j}$ has image under f_0 in the plane P_i, for $i = 1, 2$. For technical reasons (in the proof of Theorem 8.4), we also make the following further assumption: at least one endpoint of one smooth arc of ∂D is mapped by f_0 to a finite point in $\partial f_0 \subset \mathbb{R}^{2,1}$, i.e., is not an end of f_0.

Let $f_0(\lambda)$ be a smooth family of maxfaces in $\mathbb{R}^{2,1}$ depending on a parameter λ, where λ is contained in an open subset N of $\mathbb{R}^{k_1+k_2-2}$ such that $f_0(\lambda_0) = f_0$ for some $\lambda_0 \in N$. Thus, for each λ, $f_0(\lambda)$ is determined by Weierstrass data $G(\lambda), \omega(\lambda)$ and domain $D(\lambda)$ depending smoothly on λ. Assume that for each λ, we can identify the boundary arcs $S_{i,j}(\lambda)$ of $D(\lambda)$ with $S_{i,j}$ and that $f_0(\lambda)|_{S_{i,j}(\lambda)}$ is a planar geodesic in a plane $P_{i,j}(\lambda)$ parallel to P_i.

Let

$$d_{i,j} = \text{the oriented distance between } P_{i,j}(\lambda) \text{ and } P_{i,1}(\lambda)\ .$$

Thus $d_{i,j}$ changes sign when $P_{i,j}(\lambda)$ crosses from one side of $P_{i,1}(\lambda)$ to the other, and is zero if and only if $P_{i,j}(\lambda) = P_{i,1}(\lambda)$.

Definition 8.3. *$f_0(\lambda)$ is said to be* non-degenerate *with respect to the parameter λ if the period map*

$$Per : N \to (d_{1,2}, ..., d_{1,k_1}, d_{2,2}, ..., d_{2,k_2})$$

is an open map at λ_0, i.e., there exists an open neighbourhood of λ_0, $V \subset N$, such that $Per(V)$ is an open neighbourhood of the origin in $\mathbb{R}^{k_1+k_2-2}$.

We are now in a position to state the main theoretical tool of this chapter. Note that CMC-1 faces are defined in [45].

Theorem 8.4. *If $f_0(\lambda)$ is a non-degenerate maxface, then there exists a corresponding* 1-*parameter family of CMC*-1 *faces f_c, $c \in (-\epsilon, \epsilon) \setminus \{0\}$, in $\mathbb{S}^{2,1}$ with no boundary and with the same topology and corresponding reflection symmetries as f_0.*

The following argument was constructed by W. Rossman:

Sketch proof of Theorem 8.4: The proof of Theorem 8.4 is essentially the same as part of the proof of Theorem B in [47], and is the SU(1, 1) analogue of the proof using SU(2) of Theorem 5.10 in [62]. In fact some of the technicalities of that proof are not needed here because Theorem B dealt with a degenerate period problem, which differs from our case. As noted before the

Hopf differential cQ satisfies a reality condition which amounts to $\overline{Q \circ \mu_{ij}} = Q$, where $\mu_{i,j}$ denotes reflection of the surface $f_0(\lambda)$ across $S_{i,j}(\lambda)$. Furthermore, as in Lemma 4.9 in [47], $\overline{G \circ \mu_{ij}} = \sigma_j \star G$, where σ_j are particular 2×2 matrices and where

$$a \star h = \frac{a_{11}h + a_{12}}{a_{21}h + a_{22}},$$

for $a \in \mathrm{SL}(2, \mathbb{C})$ (where a_{st} denote the components of a) and h a holomorphic function. In fact, without loss of generality, σ_1 is the identity matrix and σ_2 is a unitary diagonal matrix. We then have that the solution F_c of Equation (8.1) satisfies

$$\overline{F_c \circ \mu_{i,j}} = \sigma_i F_c \rho_{i,j}^{-1} \ ,$$

where $\rho_{i,j}$ is independent of z, but can depend on c and λ, and also on the initial condition used to determine the solution F_c.

We wish to transform the $\rho_{i,j}$ so that they lie in $\mathrm{SU}(1, 1)$, because this is the condition that causes the surface f_c to have the same topology as $\hat{f}_0$. We do this as follows: we change F_c to $\hat{F}_c = F_c b$ for some $b \in \mathrm{SL}(2, \mathbb{R})$, independent of z (but allowed to depend on c and λ). Then the matrices $\rho_{i,j}$ change to $\overline{b^{-1}}\rho_{i,j} b$. We adjust b and λ until $\overline{b^{-1}}\rho_{i,j} b \in \mathrm{SU}(1, 1)$ for all i, j, for any c sufficiently close to 0, by using the non-degeneracy condition and Lemma 4.4 in [47], and arguments regarding the λ dependence of the $\rho_{i,j}$ like in [62]. This produces a one parameter family of CMC-1 surfaces $\hat{F}_c e_3 \hat{F}_c^*$ in $\mathbb{S}^{2,1}$ with the same topology as f_0 for c sufficiently close to 0, completing the proof of Theorem 8.4.

Remark 8.5. *The method in Proposition 5.4 of [72] shows the existence of a Chen-Gackstatter type maxface in $\mathbb{R}^{2,1}$, as a companion surface to the Chen-Gackstatter surface in $\mathbb{R}^3$. However, our result here will not apply to this and other companion surfaces in $\mathbb{R}^{2,1}$, because that method involves changing the G in the Weierstrass data to iG, and while the Hopf differential is real along boundary curves in the case of $\mathbb{R}^3$, it becomes pure imaginary in $\mathbb{R}^{2,1}$. Thus our result cannot be applied.*

8.3 Application: Genus-s examples with two or three ends

We now seek examples of maximal surfaces in $\mathbb{R}^{2,1}$ to which we can apply Theorem 8.4. For any positive odd number s, define a Riemann surface

$$\hat{M} := \{(z, w) \in (\mathbb{C} \cup \{\infty\})^2 \,|\, w^{s+1} = (z - \lambda_1)(z^2 - 1)^s\}, \tag{8.2}$$

of genus s with Weierstrass data

$$G = \lambda_2 z^j w^k \quad \text{and} \quad \omega = z^l w^m dz, \tag{8.3}$$

where $\lambda := (\lambda_1, \lambda_2) \in (-1, 1) \times \mathbb{R}$ and $j, k, l, m \in \mathbb{Z}$. This determines a (possibly branched) maxface

$$f := \operatorname{Re} \int_{z_0}^{z} (-2G, 1 + G^2, i - iG^2)\omega$$

on the universal cover of $M := \hat{M} \backslash \{p_1, ..., p_r\}$ in $\mathbb{R}^{2,1}$ with metric

$$ds^2 = (1 - G\bar{G})^2 \omega \bar{\omega},$$

where ds^2 blows up at the points p_j, for $j \in \{1, ..., r\}$. Consider the sheets of $\{(z, w) \in \hat{M} | \operatorname{Im} z \geq 0\}$. Let D be the sheet of $\{(z, w) \in \hat{M} | \operatorname{Im} z \geq 0\}$ such that $w((1, \infty)) \subset \mathbb{R}^+$. Label the boundary curves of D as

$$\begin{aligned}
S_{1,1} &:= \{(z, w(z)) \in D |\, z \in [1, \infty]\}, \\
S_{1,2} &:= \{(z, w(z)) \in D |\, z \in [-1, \lambda_1]\}, \\
S_{2,1} &:= \{(z, w(z)) \in D |\, z \in [\lambda_1, 1]\}, \\
S_{2,2} &:= \{(z, w(z)) \in D |\, z \in [-\infty, -1]\}.
\end{aligned}$$

To obtain surfaces possessing the reflectional symmetry that Theorem 8.4 requires, we use the following lemma:

Lemma 8.6. *The images of the boundary arcs of D are planar geodesics if and only if $k + m$ is an integer multiple of $s + 1$.*

Proof. The result follows by checking when the Hopf differential $Q = dG\,\omega$ is real valued along the boundary of D. $\square$

From now on we will assume that $k + m$ is an integer multiple of $s + 1$.

Lemma 8.7. *When $\lambda_1 = 0$ we have that $d_{2,2} = \pm d_{1,2}$.*

Proof. Define two curves in M by

$$\begin{aligned}
\tau_1, \tau_2 : [0, \pi] \to M, \quad & \tau_1(\nu) = (e^{i\nu} + \tfrac{1}{2}, w(e^{i\nu} + \tfrac{1}{2})), \text{ in } D, \text{ and} \\
& \tau_2(\nu) = (-e^{i\nu} - \tfrac{1}{2}, (-1)^{\frac{s}{s+1}} w(e^{i\nu} + \tfrac{1}{2})).
\end{aligned}$$

Note that τ_2 is well-defined in M because s is odd. Then for $\theta = \frac{\pi k s}{s+1}$,

$$\begin{aligned}
d_{1,2} &= \operatorname{Re} \int_{\tau_1} i(1 - G^2)\omega \\
d_{2,2} &= -\sin\theta \operatorname{Re} \int_{\tau_2} (1 + G^2)\omega + \cos\theta \operatorname{Re} \int_{\tau_2} i(1 - G^2)\omega
\end{aligned}$$

Thus,

$$\begin{aligned} d_{2,2} &= \mathrm{Re} \int_{\tau_2} (ie^{i\theta}\omega - ie^{-i\theta}G^2\omega) \\ &= \mathrm{Re} \int_{\tau_1} (ie^{i\theta}(-1)^{\frac{ms}{s+1}+l+1}\omega - ie^{-i\theta}(-1)^{\frac{(2k+m)s}{s+1}+l+1}G^2\omega) \\ &= (-1)^{\frac{(k+m)s}{s+1}+l+1} d_{1,2}\,, \end{aligned}$$

since $k+m$ is an integer multiple of $s+1$. □

In light of Lemma 8.7, we set $\lambda_1 = 0$ and our goal is to find a value of λ_2 so that $d_{1,2} = 0$, i.e., so that our surface is well-defined on M and thus has finite topology. Viewing $d_{1,2}$ as a function of λ_2, one arrives at the following lemma:

Lemma 8.8. *If*

$$\lambda_2^{\pm} := \pm\sqrt{\frac{Im \int_\tau z^l w^m dz}{Im \int_\tau z^{2j+l} w^{2k+m} dz}}$$

are real and non-zero, where

$$\tau : [0,\pi] \to \Sigma, \quad \nu \mapsto (e^{i\nu} + \tfrac{1}{2}, w(e^{i\nu} + \tfrac{1}{2})),$$

then the maximal surfaces determined by (8.2) *and* (8.3) *with* $\lambda = \lambda_0^{\pm} := (0, \lambda_2^{\pm})$ *are well-defined on* M.

Therefore let us assume that $\lambda_0 := (0, \lambda_2^0)$ determines a maximal surface that is well defined on M. We want the surface to be a maxface, i.e., we want to allow the surface to admit singularities but have no branch points and to have complete ends. Away from points of M where z is not a local coordinate, i.e., when $z \in \{0, 1, -1, \infty\}$, we have that df is non-zero, and thus, the surface is not branched. Furthermore, the surface is a maxface away from $z \in \{0, 1, -1, \infty\}$, which follows from Fact 1.1 in [47], considered on local simply connected open subsets of M.

To ensure that the surface does not have branch points when $z \in \{0, 1, -1, \infty\}$, we will require that the metric

$$ds^2 = (1 - G\bar{G})^2\omega\bar{\omega} = (1 - \lambda_2^2|z^j w^k|^2)^2 |z^l w^m|^2 |dz|^2$$

is either non-singular or blows up at these points:

At $z = 0$, assuming that $j(s+1) + k \neq 0$, the metric is non-singular or blows up if and only if

$$\begin{aligned} l(s+1) + m + s \le 0 \quad &\text{when} \quad j(s+1) + k > 0\,, \\ (2j + l + 1)(s+1) + 2k + m - 1 \le 0 \quad &\text{when} \quad j(s+1) + k < 0. \end{aligned}$$

Equality on the left hand side in either case means that f has a finite point at $z = 0$. Otherwise,

f admits a complete end at $z = 0$.

At $z = \pm 1$, assuming $k \neq 0$, the metric is non-singular or blows up if and only if

$$\begin{aligned} m + 1 &\leq 0 \quad \text{when} \quad k > 0\,, \\ 2k + m + 1 &\leq 0 \quad \text{when} \quad k < 0. \end{aligned}$$

If we have equality on the left hand side in either case then f has finite points at $z = \pm 1$. Otherwise, $z = \pm 1$ are both complete ends of the surface.

At $z = \infty$, assuming $(2k + j)s + j + k \neq 0$, the metric is non-singular or blows up if and only if

$$\begin{aligned} (l + 2m + 1)s + l + m + 2 &\geq 0 \quad \text{when} \quad (2k + j)s + j + k < 0\,, \\ (4k + 2j + l + 2m + 1)s + 2j + 2k + l + m + 2 &\geq 0 \quad \text{when} \quad (2k + j)s + j + k > 0. \end{aligned}$$

If we have equality on the left hand side in either case then f has a finite point at $z = \infty$. Otherwise, f has a complete end at $z = \infty$.

So far we have shown how to construct maxfaces in $\mathbb{R}^{2,1}$ with finite topology equal to that of M. We would now like to use Theorem 8.4 to obtain CMC 1 faces in $\mathbb{S}^{2,1}$ from these examples. To do this we create a period problem by viewing $\lambda = (\lambda_1, \lambda_2)$ in (8.2) and (8.3) as a parameter in the domain $(-1, 1) \times \mathbb{R}$. Then to check that the period problem is non-degenerate, it suffices to check that the map

$$(\lambda_1, \lambda_2) \mapsto (d_{1,2}, d_{2,2})$$

is immersed at the solution point λ_0 of the period problem, i.e., the determinant of the Jacobian at λ_0,

$$\left.\begin{pmatrix} \frac{\partial}{\partial \lambda_1} d_{1,2} & \frac{\partial}{\partial \lambda_2} d_{1,2} \\ \frac{\partial}{\partial \lambda_1} d_{2,2} & \frac{\partial}{\partial \lambda_2} d_{2,2} \end{pmatrix}\right|_{\lambda = \lambda_0},$$

is non-zero. One can check that

$$\begin{aligned} \tfrac{\partial}{\partial \lambda_1} d_{2,2}|_{\lambda=\lambda_0} &= (-1)^{l + \frac{(k+m)s}{s+1}} \tfrac{\partial}{\partial \lambda_1} d_{1,2}|_{\lambda=\lambda_0}, \text{ and} \\ \tfrac{\partial}{\partial \lambda_2} d_{2,2}|_{\lambda=\lambda_0} &= (-1)^{l + 1 + \frac{(k+m)s}{s+1}} \tfrac{\partial}{\partial \lambda_2} d_{1,2}|_{\lambda=\lambda_0}. \end{aligned}$$

Thus, the non-degeneracy condition reduces to both $\frac{\partial}{\partial \lambda_1} d_{1,2}|_{\lambda=\lambda_0}$ and $\frac{\partial}{\partial \lambda_2} d_{1,2}|_{\lambda=\lambda_0}$ being non-zero. This is the case in all the examples we are about to consider.

Remark 8.9. *If introducing parameters λ_j, like above, in a certain way does not yield a non-degenerate period problem for a particular example, then there may be other ways that parameters can be introduced into the data so that a non-degenerate period problem is obtained.*

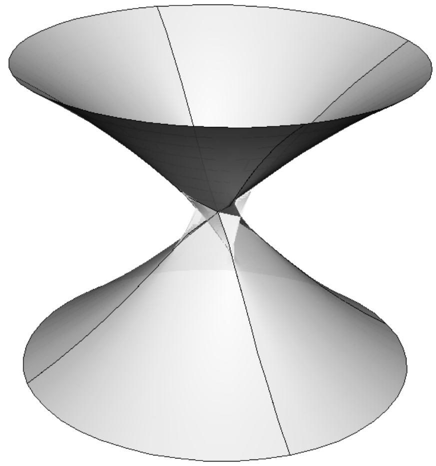

Figure 8-1: Genus 1 example with $j = -1$, $k = 1$, $l = 0$, $m = -1$, see [47].

8.3.1 Two-ended examples

If we consider the case where

$$j = -1,\ k = 1,\ l = 0,\ m = -1,$$

then we obtain the genus s surfaces with two complete ends in $\mathbb{R}^{2,1}$ given in [47]. $(z, w) = (0, 0)$ and $(z, w) = (\infty, \infty)$ correspond to the ends. When $s = 1$, these ends are embedded. Figure 8-1 shows the genus 1 case. In [47], a period problem that is not non-degenerate is considered for these surfaces. However, if we introduce parameters (λ_1, λ_2) as above, the period problem becomes non-degenerate and we can apply Theorem 8.4.

We also give two new examples of genus s maxfaces with two complete ends in $\mathbb{R}^{2,1}$ which when parameters are introduced as above have non-degenerate period problems, see Figures 8-2 and 8-3.

8.3.2 Three-ended example

Problem 2 in [47] asked whether there are maxfaces of positive genus in $\mathbb{R}^{2,1}$ and $\mathbb{S}^{2,1}$ with more than two complete ends. We give an affirmative answer to this question by taking $s = 1$ and

$$j = 2,\ k = -1,\ l = 0,\ m = -1.$$

This determines a genus 1 maxface with three complete embedded ends in $\mathbb{R}^{2,1}$, see Figure 8-4, and by introducing parameters as above we have a non-degenerate period problem and can thus obtain a corresponding one-parameter family of surfaces in $\mathbb{S}^{2,1}$.

We also give another example of genus $s = 1$ maxface with three complete ends in $\mathbb{R}^{2,1}$ which when parameters are introduced as above has a non-degenerate period problems, see Figure 8-5.

Figure 8-2: Genus 1 example with two complete ends with $j = 0$, $k = 1$, $l = -1$, $m = -1$. $(z, w) = (0, 0)$ corresponds to an embedded end, and $(z, w) = (\infty, \infty)$ corresponds to a non-embedded end.

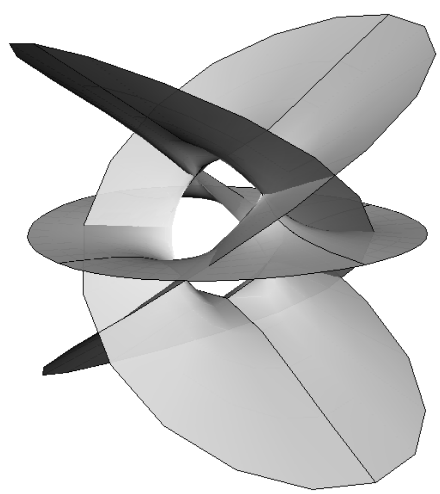

Figure 8-3: Genus 1 example with two complete ends with $j = 0$, $k = 1$, $l = -2$, $m = -1$. $(z, w) = (0, 0)$ and $(z, w) = (\infty, \infty)$ correspond to the ends. When $s = 1$, the end corresponding to $(z, w) = (\infty, \infty)$ is embedded.

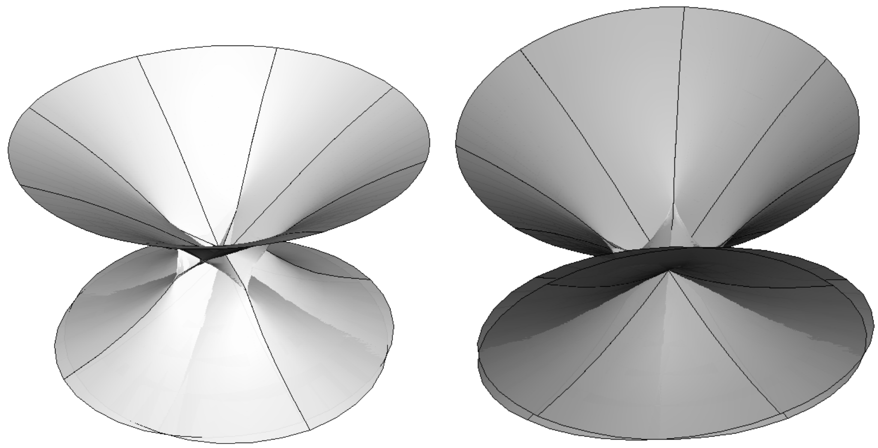

Figure 8-4: Two different views of the genus 1 example with three complete embedded ends with $j = 2$, $k = -1$, $l = 0$, $m = -1$. $(z, w) = (\infty, \infty)$ corresponds to the end on the top of the figure, and $(z, w) = (\pm 1, 0)$ correspond to the ends on the bottom of the figure.

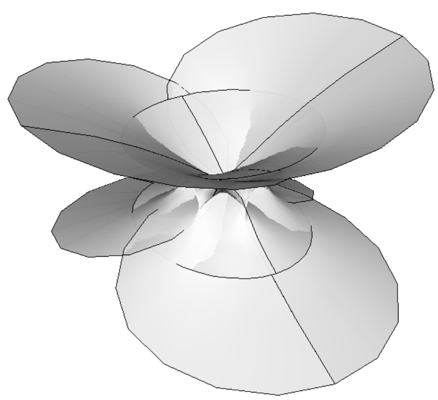

Figure 8-5: Genus 1 example with three complete ends with $j = 3$, $k = -1$, $l = 0$, $m = -1$. $(z, w) = (\pm 1, 0)$ correspond to embedded ends and $(z, w) = (\infty, \infty)$ corresponds to non-embedded ends.

Remark 8.10. *In this chapter we have only given examples of maxfaces of odd genus, but using the framework of Section 4, we believe it should be possible to construct examples of maxfaces in* $\mathbb{R}^{2,1}$ *(and CMC-*1 *faces in* $\mathbb{S}^{2,1}$*) with arbitrary genus and two or three ends.*

Chapter 9

Conclusion

In conclusion we shall present some open questions and further work that arises from this thesis.

- In Chapter 3 we give results regarding deformations of maps into submanifolds of projective space. We then proceed to apply these results to projective, conformal and Lie geometry. These are all examples of R-spaces. Since R-spaces can be represented as submanifolds of projective space using the highest weight representation, we can apply our result in this more general setting. It would be interesting to see how the properties of R-spaces, e.g. their height, affects the deformability of maps into these spaces.

- Remark 4.1 states that it is possible for a Legendre map to be Lie-applicable in multiple ways. The case for which they can be applicable in 3-parameters worth of ways has been studied in [43, 56]. It would be interesting to see how this multi-applicability can be characterised, especially in the 2-parameter case.

- In Section 4.3 we treat the associate surfaces related to an Ω-surface. There is some mystery surrounding the nature of the associate Gauss map that we defined. Furthermore, since one of the invariants of Laguerre geometry is parallel planes it seems that this is the appropriate setting for studying these surfaces. Another question that arises from this section is whether anything interesting can be said of quadruples of O-surfaces $\{\nu^a, \nu^b, \nu^c, \nu^d\}$ whose principal curvatures satisfy the relation

$$\frac{1}{\kappa_1^a \kappa_2^b} + \frac{1}{\kappa_2^a \kappa_1^b} - \frac{1}{\kappa_1^c \kappa_2^d} - \frac{1}{\kappa_1^d \kappa_2^c} = 0.$$

- In [33], Darboux seeks a characterisation of pairs of surfaces that are Ribaucour transforms of each other and induce the same conformal structure. It turns out that pairs of isothermic surfaces that are Darboux transforms of each other are the only non-trivial pairings with this property. For a modern account of this see [50, Chapter 3]. It would be interesting to see if one can develop a similar problem for which pairs of Ω-surfaces that are Darboux transforms of each other are the only non-trivial solutions.

- In [15, Section 4] it is shown that linear Weingarten surfaces that admit a Weierstrass-type representation are L-isothermic. Theorem 6.34 gives a characterisation of L-isothermic surfaces as those Ω-surfaces that admit a totally umbilic Darboux transform. This suggests that Weierstrass-type representations can be described in terms of this Darboux transform. This would unify all the Weierstrass-type representations and give a geometric interpretation for these representations. Furthermore, one can ask whether general L-isothermic surfaces have such a representation.

Bibliography

[1] L. Bianchi. Complementi alle ricerche sulle superficie isoterme. *Ann. di Mat.*, 12:19–54, 1905.

[2] L. Bianchi. Ricerche sulle superficie isoterme e sulla deformazione delle quadriche. *Ann. di Mat.*, 11:93–157, 1905.

[3] W. Blaschke. *Vorlesungen über Differentialgeometrie III.* Springer Grundlehren XXIX, Berlin, 1929.

[4] F. Burstall, D. Ferus, F. Pedit, and U. Pinkall. Harmonic tori in symmetric spaces and commuting Hamiltonian systems on loop algebras. *Ann. of Math. (2)*, 138(1):173–212, 1993.

[5] F. E. Burstall. Isothermic surfaces: conformal geometry, Clifford algebras and integrable systems. In *Integrable systems, geometry, and topology*, volume 36 of *AMS/IP Stud. Adv. Math.*, pages 1–82. Amer. Math. Soc., Providence, RI, 2006.

[6] F. E. Burstall and D. M. J. Calderbank. Conformal submanifold geometry iv-v. Work in progress.

[7] F. E. Burstall and D. M. J. Calderbank. Submanifold geometry in generalized flag manifolds. *Rend. Circ. Mat. Palermo (2) Suppl.*, (72):13–41, 2004.

[8] F. E. Burstall and D. M. J. Calderbank. Conformal submanifold geometry i-iii. *ArXiv e-prints*, June 2010.

[9] F. E. Burstall and D. M. J. Calderbank. Private communication. 2010.

[10] F. E. Burstall and D. M. J. Calderbank. Notes taken from "Integrability in projective differential geometry" course. 2014.

[11] F. E. Burstall, N. M. Donaldson, F. Pedit, and U. Pinkall. Isothermic submanifolds of symmetric R-spaces. *J. Reine Angew. Math.*, 660:191–243, 2011.

[12] F. E. Burstall and U. Hertrich-Jeromin. Harmonic maps in unfashionable geometries. *Manuscripta Math.*, 108(2):171–189, 2002.

[13] F. E. Burstall and U. Hertrich-Jeromin. The Ribaucour transformation in Lie sphere geometry. *Differential Geom. Appl.*, 24(5):503–520, 2006.

[14] F. E. Burstall, U. Hertrich-Jeromin, and W. Rossman. Lie geometry of flat fronts in hyperbolic space. *C. R. Math. Acad. Sci. Paris*, 348(11-12):661–664, 2010.

[15] F. E. Burstall, U. Hertrich-Jeromin, and W. Rossman. Lie geometry of linear Weingarten surfaces. *C. R. Math. Acad. Sci. Paris*, 350(7-8):413–416, 2012.

[16] F. E. Burstall, U. Hertrich-Jeromin, and W. Rossman. Discrete linear Weingarten surfaces. *ArXiv e-prints*, June 2014.

[17] F. E. Burstall and J. H. Rawnsley. *Twistor theory for Riemannian symmetric spaces*, volume 1424 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1990. With applications to harmonic maps of Riemann surfaces.

[18] F. E. Burstall and S. D. Santos. Special isothermic surfaces of type d. *J. Lond. Math. Soc.*, 85(2):571–591, 2012.

[19] P. Calapso. Sulle superficie a linee di curvatura isoterme. *Rend. Circ. Mat. Palermo*, 17:275–286, 1903.

[20] P. Calapso. Alcune superficie di guichard e le relative trasformazioni. *Ann. di Mat.*, 11:201–251, 1905.

[21] D. M. J. Calderbank. Omega surfaces and Guichard surfaces. Talk given at Bath geometry seminar, 2010.

[22] E. Cartan. Sur la déformation projective des surfaces. *Ann. Sci. École Norm. Sup. (3)*, 37:259–356, 1920.

[23] E. Cartan. Sur le probléme général de la déformation. *C. R. Congrés Strasbourg*, pages 397–406, 1920.

[24] E. Cartan. Les espaces à connexion conforme. *Ann. Soc. Pol. Math.*, 2:171–221, 1923.

[25] T. E. Cecil. *Lie sphere geometry*. Universitext. Springer, New York, second edition, 2008. With applications to submanifolds.

[26] E. B. Christoffel. Ueber einige allgemeine eigenschaften der minimumsflächen. *Crelle's J*, 67:218–228, 1867.

[27] D. J. Clarke. Moebius-flat submanifolds I and II. Talks given at Bath geometry seminar, 2011.

[28] D. J. Clarke. *Integrability in submanifold geometry*. PhD thesis, University of Bath, 2012.

[29] D. J. Clarke. Möbius-flat hypersurfaces in projective space. *ArXiv e-prints*, March 2012.

[30] M. Dajczer and R. Tojeiro. An extension of the classical Ribaucour transformation. *Proc. London Math. Soc. (3)*, 85(1):211–232, 2002.

[31] M. Dajczer and R. Tojeiro. Commuting Codazzi tensors and the Ribaucour transformation for submanifolds. *Results Math.*, 44(3-4):258–278, 2003.

[32] G. Darboux. Sur les surfaces isothermiques. *Ann. Ec. Norm. Sup.*, 16:491–508, 1899.

[33] G. Darboux. Sur les surfaces isothermiques. *C.R. Acad. Sci. Paris*, 128:1299–1305, 1899.

[34] A. Demoulin. Sur les surfaces Ω. *C. R. Acad. Sci. Paris*, 153:927–929, 1911.

[35] A. Demoulin. Sur les surfaces R et les surfaces Ω. *C. R. Acad. Sci. Paris*, 153:590–593, 1911.

[36] A. Demoulin. Sur les surfaces R et les surfaces Ω. *C. R. Acad. Sci. Paris*, 153:705–707, 1911.

[37] L. P. Eisenhart. Transformations of surfaces of Guichard and surfaces applicable to quadrics. *Annali di Mat.*, 22(1):191–247, 1914.

[38] L. P. Eisenhart. Surfaces Ω and their transformations. *Trans. Amer. Math. Soc.*, 16(3):275–310, 1915.

[39] L. P. Eisenhart. Transformations of surfaces Ω. II. *Trans. Amer. Math. Soc.*, 17(1):53–99, 1916.

[40] L. P. Eisenhart. *Transformations of surfaces.* Second edition. Chelsea Publishing Co., New York, 1962.

[41] E. V. Ferapontov. Integrable systems in projective differential geometry. *Kyushu J. Math.*, 54(1):183–215, 2000.

[42] E. V. Ferapontov. Lie sphere geometry and integrable systems. *Tohoku Math. J. (2)*, 52(2):199–233, 2000.

[43] E. V. Ferapontov. The analogue of Wilczynski's projective frame in Lie sphere geometry: Lie-applicable surfaces and commuting Schrödinger operators with magnetic fields. *Internat. J. Math.*, 13(9):959–985, 2002.

[44] G. Fubini. Applicabilità projettiva di due superficie. *Rend. Circ. Mat. Palermo*, 41:135–162, 1916.

[45] S. Fujimori. Spacelike CMC 1 surfaces with elliptic ends in de Sitter 3-space. *Hokkaido Math. J.*, 35(2):289–320, 2006.

[46] S. Fujimori, S. Gaber, and M. Pember. Maximal surfaces in Minkowski 3-space with non-trivial topology and corresponding CMC 1 surfaces in de Sitter 3-space. To appear in Kobe J. Math.

[47] S. Fujimori, W. Rossman, M. Umehara, K. Yamada, and S-D. Yang. New maximal surfaces in Minkowski 3-space with arbitrary genus and their cousins in de Sitter 3-space. *Results Math.*, 56(1-4):41–82, 2009.

[48] P. Griffiths. On Cartan's method of Lie groups and moving frames as applied to uniqueness and existence questions in differential geometry. *Duke Math. J.*, 41:775–814, 1974.

[49] C. Guichard. Sur les surfaces isothermiques. *C. R. Math. Acad. Sci. Paris*, 130:159–162, 1900.

[50] U. Hertrich-Jeromin. *Introduction to Möbius differential geometry*, volume 300 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2003.

[51] G. R. Jensen. Deformation of submanifolds of homogeneous spaces. *J. Differential Geom.*, 16(2):213–246, 1981.

[52] S. Lie. Über Komplexe, insbesondere Linien-und Kugelkomplexe, mit Anwendung auf der Theorie der partieller Differentialgleichungen. *Math. Ann.*, 5:145–208, 209–256, 1872.

[53] E. Musso and L. Nicolodi. A variational problem for surfaces in Laguerre geometry. *Trans. Amer. Math. Soc.*, 348:4321–4337, 1996.

[54] E. Musso and L. Nicolodi. Isothermal surfaces in Laguerre geometry. *Boll. Un. Mat. Ital. B (7)*, 11:125–144, 1997.

[55] E. Musso and L. Nicolodi. The Bianchi-Darboux transform of L-isothermic surfaces. *Internat. J. Math.*, 11(7):911–924, 2000.

[56] E. Musso and L. Nicolodi. Deformation and applicability of surfaces in Lie sphere geometry. *Tohoku Math. J. (2)*, 58(2):161–187, 2006.

[57] E. Musso and L. Nicolodi. Holomorphic differentials and Laguerre deformation of surfaces. *ArXiv e-prints*, January 2014.

[58] B. O'Neill. *Semi-Riemannian Geometry With Applications to Relativity*, volume 103 of *Pure and Applied Mathematics*. Elsevier Science, 1983.

[59] U. Pinkall. Dupin hypersurfaces. *Math. Ann.*, 270(3):427–440, 1985.

[60] W. Rossman. Lie sphere geometry and Omega surfaces I and II. Talks given at Bath geometry seminar, 2010.

[61] W. Rossman. Isothermic surfaces in Möbius and Lie sphere geometries. Unpublished notes, 2014.

[62] W. Rossman, M. Umehara, and K. Yamada. Irreducible constant mean curvature 1 surfaces in hyperbolic space with positive genus. *Tohoku Math. J. (2)*, 49(4):449–484, 1997.

[63] P. J. Ryan. Homogeneity and some curvature conditions for hypersurfaces. *Tôhoku Math. J. (2)*, 21:363–388, 1969.

[64] S. Santos. *Special isothermic surfaces.* PhD thesis, University of Bath, 2008.

[65] W. K. Schief and B. G. Konopelchenko. On the unification of classical and novel integrable surfaces. I. Differential geometry. *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.*, 459(2029):67–84, 2003.

[66] W. K. Schief, A. Szereszewski, and C. Rogers. On shell membranes of Enneper type: generalized Dupin cyclides. *J. Phys. A*, 42(40):404016–404033, 2009.

[67] A. Szereszewski. L-isothermic and L-minimal surfaces. *J. Phys. A*, 42(11):115203–115217, 2009.

[68] K. Tenenblat. On Ribaucour transformations and applications to linear Weingarten surfaces. *An. Acad. Brasil. Ciênc.*, 74(4):559–575, 2002.

[69] Chuu-Lian Terng and Karen Uhlenbeck. Bäcklund transformations and loop group actions. *Comm. Pure Appl. Math.*, 53(1):1–75, 2000.

[70] Chuu-Lian Terng and Karen Uhlenbeck. Geometry of solitons. *Notices Amer. Math. Soc.*, 47(1):17–25, 2000.

[71] J. Tits. Sur les R-espaces. *C. R. Acad. Sci. Paris*, 239:850–852, 1954.

[72] M. Umehara and K. Yamada. Maximal surfaces with singularities in Minkowski space. *Hokkaido Math. J.*, 35(1):13–40, 2006.